CONJECTURES IN MATHEMATICS

Generalization is a very important tool for obtaining new results in mathematics. We first experiments with numbers and geometrical shapes and observe some patterns emerging and then we make some conjectures on the basis of these observations. These conjectures do not get the status of theorems unless these can be deducted either from certain axioms or can be deduced from other results which have been earlier proved in mathematics.

Let us consider some examples of valid as well as invalid generalizations.

CONJECTURES BASED ON PRIME NUMBERS:

(1) An expression proposed for giving prime numbers only was $f(n) = n^2 - n + 41$. Its values are given below:

n	1	2	3	4	5	6	7	8	9	10…
$f(n)$	41	43	47	53	61	71	83	97	113	131…

We will observe that for values of n up to 40, the expression always gives a prime number. But for n = 41, it gives 41^2 which is a composite number. Thus the conjecture is invalid.

(2) Another expression which for more than a century was believed to give prime numbers only was $F_n = 2^{2^n} + 1$.

Observe that $F_1 = 5$, $F_2 = 17$, $F_3 = 257$, $F_4 = 63537$. But when n = 5, $F_5 = 4234367297$, which is a composite number as it has 641 as its factor, other factor being 6700417.

Thus, through the first four Fermat numbers are prime, but not the fifth number is prime.

(3) We know that 2, 3, 5, 7, 11, 13, 17,… are prime numbers. We also observe :

$$2+1=3$$
$$2\times3+1=7$$
$$2\times3\times5+1=31$$
$$2\times3\times5\times7+1=211$$
$$2\times3\times5\times7\times9+1=2311$$

All these numbers are prime numbers. We may thus conjecture that if we multiply the first *n* consecutive prime numbers and add 1, we shall always get a prime number.

The conjecture is however easily seen to be false, since the next number is $2\times3\times5\times7\times9\times11+1=59\times509$, which is composite number.

(4) Write 25 natural numbers in five rows of five numbers each.

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25

We find that each row contains at least one prime number. Next if we write first 36 natural numbers in six rows of six numbers each, we again find each row contains at least one prime number. (Verify!)

So we may conjecture that if we write the first n^2 natural numbers in n rows of *n* natural numbers each, each row will contain at least one prime number.

This conjecture has been verified or proved for all values of *n* up to 4500. Yet we cannot say that the result is true for all values of *n*, since no proof for the general result is so far known.

(5) $n! - (n-1)! + (n-2)! + ... + (-1)^n$ is a prime number. We find:

$3! - 2! + 1! = 5$

$4! - 3! + 2! - 1! = 19$

$5! - 4! + 3! - 2! + 1! = 101$

$6! - 5! + 4! - 3! + 2! - 1! = 619$

$7! - 6! + 5! - 4! + 3! - 2! + 1! = 4421$

$8! - 7! + 6! - 5! + 4! - 3! + 2! - 1! = 35899$

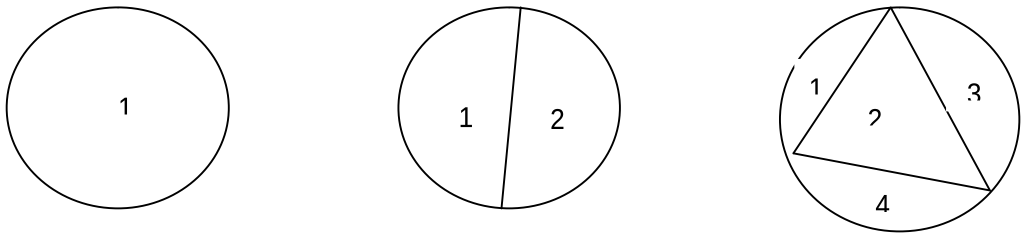
and so far the conjecture holds , but

$9! - 8! + 7! - 6! + 5! - 4! + 3! - 2! + 1! = 326981 = 79 \times 4139$

Thus, the conjecture is not valid for all numbers.

SOME MORE CONJECTURES:

(1) The maximum number of regions formed by joining n points on the circumference of a circle 2^{n-1}.



n =1, R = 1 n =2, R = 2 n =3, R =4

It is clear that, for 1 point, we get a maximum of 1 region

For 2 points, we get a maximum of 2 regions;

For 3 points, we get a maximum of 4 regions;

For 4 points, we get a maximum of 8 regions;

For 5 points, we get a maximum of 16 regions… (Draw diagrams and verify!)

All these verify the conjecture, but however hard we may try for 6 points, we get a maximum of 31 regions instead of 32 regions as suggested by the conjecture (verify!). Similarly for 7 points we get a maximum of 57 regions instead of 64 regions as suggested by the conjecture. Thus the conjecture is false. In fact the maximum number of regions is given by $1 + {}^nC_2 + {}^nC_4$, where ${}^nC_r$ represents combinations.

(2) **Fermat's Conjecture:** The Diophantine equation $x^2 + y^2 = z^2$ has an infinitely of solutions in positive integers. Fermat conjectured that $x^n + y^n = z^n$ has no solution in positive integers for any value of $n > 2$. It was a conjecture, but it was called Fermat's last theorem though it was proved only in 1996.

(3) **Four Colour Conjecture:** It states that every map in a plane can be coloured by at most 4 colours in such a way that no two adjoining countries will have the same colour. Millions of complicated maps were drawn and no map required more than 4 colours, but this did not constitute a proof. The final proof was however given in 1976 by making use of computers and thus this result is now known as Four Colour Theorem.

(4) **Problems of antiquity:** For two thousand years, mathematicians tried the following constructions with ruler and compass only:

- **(i)** **Duplicating a cube i.**e. constructing a length equal to $\sqrt[3]{2}$.
- **(ii)** **Squaring a circle** i.e. constructing a length equal to $\sqrt[3]{2}$.
- **(iii)** **Trisecting an angle** i.e. constructing an angle $\frac{\theta}{3}$ when the angle θ is given.

And they failed and the possibility of making these constructions remained a conjecture, till it was finally proved mathematically that all these constructions are impossible by using ruler and compass only.

Conjectures are obtained by generalizations but only those generalizations are valid which can be proved rigorously. We must continue to generalize, but we must generalize with care.

So, what did you conjecture today!