Synopsis – Grade 12 Math Part II

Chapter 7: Integrals

- Integration is the inverse process of differentiation. If $\frac{d}{dx}f(x)=g(x)$, then we can write $\int g(x)dx = f(x) + C$. This is called the general or the indefinite integral and C is called the constant of integration.

- **Some standard indefinite integrals**
  - $\int x^n dx=\frac{x^{n+1}}{n+1}+C, n\neq -1$
  - $\int dx = x + C$
  - $\int \sin x dx = -\cos x + C$
  - $\int \cos x dx = \sin x + C$
  - $\int \sec^2 x dx = \tan x + C$
  - $\int \text{cosec}^2 x dx = -\cot x + C$
  - $\int \sec x \tan x dx = \sec x + C$
  - $\int \text{cosec}x \cot x dx = -\text{cosec}x + C$
  - $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x + C$ or $-\cos^{-1}x + C$
  - $\int \frac{dx}{1+x^2} = \tan^{-1}x + C$ or $-\cot^{-1}x + C$
  - $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1}x + C$ or $-\text{cosec}^{-1}x + C$
  - $\int e^x dx = e^x + C$
  - $\int a^x dx = \frac{a^x}{\log a} + C$
  - $\int \frac{1}{x} dx = \log|x| + C$
  - $\int e^{ax} dx = \frac{e^{ax}}{a} + C$

- **Properties of indefinite integrals**
  - $\frac{d}{dx}\int f(x)dx = f(x)$ and $\int f'(x)dx = f(x) + C$
  - If the derivative of two indefinite integrals is the same, then they belong to same family of curves and hence they are equivalent.
  - $\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$

- $\int kf(x)dx = k\int f(x)dx$, where k is any constant

❖ **Methods of integration**

There are three important methods of integration, namely, **integration by substitution**, **integration using partial fractions**, and **integration by parts**.

❖ **Integration by substitution**

A change in the variable of integration often reduces an integral to one of the fundamental integrals, which can be easily found out. The method in which we change the variable to some other variable is called the method of substitution.

Using substitution method of integration, we obtain the following standard integrals:

- $\int \tan x dx = -\log|\cos x| + C$ or $\log|\sec x| + C$
- $\int \cot x dx = \log|\sin x| + C$
- $\int \sec x dx = \log|(\sec x + \tan x)| + C$
- $\int \operatorname{cosec} x dx = \log|\operatorname{cosec} x - \cot x| + C$

❖ **Integration by partial fractions**

The following table shows how a function of the form $\frac{P(x)}{Q(x)}$, where $Q(x) \neq$ and degree of $Q(x)$ is greater than the degree of $P(x)$, is broken by the concept of partial fractions. After doing this, we find the integration of the given function by integrating the right hand side (i.e., partial fractional form).

Function	Form of partial fraction
$\frac{px+q}{(x-a)(x-b)}, a \neq b$	$\frac{A}{x-a} + \frac{B}{x-b}$
$\frac{px+q}{(x-a)^2}$	$\frac{A}{x-a} + \frac{B}{(x-a)^2}$
$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$
$\frac{px^2+qx+r}{(x-a)^2(x-b)}$	$\frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{x-b}$
$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$, where $x^2 + bx + c$ cannot be factorised	$\frac{A}{x-a} + \frac{Bx+C}{x^2+bx+c}$

Here, A, B, C are constants that are to be determined.

❖ **Integrals of some special functions**

- $\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log\left|\frac{x-a}{x+a}\right| + \text{C}$
- $\int \frac{1}{a^2 - x^2} dx = \frac{1}{2a} \log\left|\frac{a+x}{a-x}\right| + \text{C}$
- $\int \frac{1}{\sqrt{x^2 - a^2}} dx = \log\left|x + \sqrt{x^2 - a^2}\right| + \text{C}$
- $\int \frac{1}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + \text{C}$
- $\int \frac{1}{\sqrt{x^2 + a^2}} dx = \log\left|x + \sqrt{x^2 + a^2}\right| + \text{C}$
- $\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + \text{C}$

❖ **Method of some special types of integrals**

- Integrals of the types $\int \frac{dx}{ax^2 + bx + c}$ or $\int \frac{dx}{\sqrt{ax^2 + bx + c}}$:

  We can reduce these types of integrals into standard form by expressing $ax^2 + bx + c$ as $a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right] = a\left[\left(x + \frac{b}{2a}\right)^2 + \left(\frac{c}{a} - \frac{b^2}{4a^2}\right)\right]$ and then applying substitution method by putting $x + \frac{b}{2a}$ as u (say).

- Integrals of the type $\int \frac{px+q}{ax^2 + bx + c} dx$ or $\int \frac{px+q}{\sqrt{ax^2 + bx + c}} dx$:

  These types of integrals can be transformed into standard form by expressing $px + q$ as $\text{A} \cdot \frac{d}{dx}(ax^2 + bx + c) + \text{B} = \text{A}(2ax + b) + \text{B}$, where A and B are determined by comparing coefficients on both sides.

❖ **Integration by parts**

For given functions $f(x)$ and $g(x)$,

$$\int f(x) \cdot g(x) dx = f(x) \int g(x) dx - \int \left[f'(x) \cdot \int g(x) dx \right] dx$$

In other words, the integral of the product of two functions is equal to first function × integral of the second function – integral of {differential of the first function × integral of the second function}.

Here, the functions f and g have to be taken in proper order with respect to the ILATE rule, where I, L, A, T, and E respectively represent inverse, logarithm, arithmetic, trigonometric, and exponential function.

- We can find the integrals of the type $\int e^x \left[f(x) + f'(x) \right] dx$ by using the ILATE rule and obtain $\int e^x \left[f(x) + f'(x) \right] dx = \int e^x f(x) dx + \text{C}$

- Using the method of integration by parts, we obtain the following standard integrals:

  i. $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right| + C$

  ii. $\sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right| + C$

  iii. $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$

❖ Definite integrals

- A definite integral is denoted by $\int_a^b f(x)dx$, where a is the lower limit and b is the upper limit of the integral. If $\int f(x)dx = F(x) + C$, then

  $\int_a^b f(x)dx = F(b) - F(a)$

- The definite integral $\int_a^b f(x)dx$ represents the area function A(x) since $\int_a^b f(x)dx$ is the area bounded by the curve $y = f(x)$, $x \in [a, b]$, the x-axis, and the ordinates $x = a$ and $x = b$

- The definite integral $\int_a^b f(x)dx$ can be expressed as the sum of limits as

  $\int_a^b f(x)dx = (b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a) + f(a+h) + \ldots + f(a(n-1)h)\right]$,where

  $h = \frac{b-a}{n} \to 0$ as $n \to \infty$

❖ First fundamental theorem of integral calculus

Let f be a continuous function on the closed interval [a, b] and let A (x) be the area function. Then, $A'(x) = f(x)\ \forall x \in [a,b]$

❖ Second fundamental theorem of integral calculus

Let f be a continuous function on the closed interval [a, b] and let F be an anti-derivative of f. Then,

$$\int_a^b f(x)dx = \left[F(x)\right]_a^b = F(b) - F(a)$$

❖ Some useful properties of definite integrals

- $\int_a^b f(x)dx = \int_a^b f(t)dt$

- $\int_a^b f(x)dx = -\int_b^a f(x)dx \cdot$ In particular, $\int_a^a f(x)dx = 0$
- $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$
- $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$
- $\int_0^a f(x)dx = \int_0^a f(a-x)dx$
- $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$
- $\int_0^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, \text{ if } f(2a-x) = f(x) \\ 0, \text{ if } f(2a-x) = -f(x) \end{cases}$
- $\int_{-a}^{a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, \text{ if } f \text{ is an even function i.e., if } f(-x) = f(x) \\ 0, \text{ if } f \text{ is an odd function i.e., if } f(-x) = -f(x) \end{cases}$

Chapter 8: Application of Integrals

❖ Area under simple curves

- Area of the region bounded by the curve $y = f(x)$, x-axis, and the lines $x = a$ and $x = b$ $(b > a)$ is given by $A = \int_a^b y\,dx$ or $A = \int_a^b f(x)\,dx$
- The area of the region bounded by the curve $x = g(y)$, y-axis, and the lines $y = c$ and $y = d$ is given by $A = \int_c^d x\,dy$ or $A = \int_c^d g(y)\,dx$

❖ Area of the region bounded by a curve and a line

- If a line $y = mx + p$ intersects a curve $y = f(x)$ at a and b, then the area of this curve under the line $y = mx + p$ or the lines $x = a$ and $x = b$ is

  $$A = \int_a^b y\,dx \text{ or } A = \int_a^b f(x)\,dx$$

- If a line $y = mx + p$ intersects a curve $x = g(y)$ at c and d respectively, then area of this curve under the line $y = mx + p$ or lines $y = c$ and $y = d$ is given by,

  $$A = \int_c^d x\,dy = \int_c^d g(y)\,dy$$

❖ Area between two curves

The area of the region enclosed between two curves $y = f(x)$ and $y = g(x)$ and the lines $x = a$ and $x = b$ is given by,

$$A = \begin{cases} \int_a^b [f(x) - g(x)]dx, \text{ where } f(x) \geq g(x) \text{ in } [a, b] \\ \int_a^c [f(x) - g(x)]dx + \int_c^b [g(x) - f(x)]dx, \\ \text{where } a < c < b \text{ and } f(x) \geq g(x) \text{ in } [a, c] \text{ and } f(x) \leq g(x) \text{ in } [c, b] \end{cases}$$

Chapter 9: Differential Equations

- An equation is called a differential equation, if it involves variables as well as derivatives of dependent variable with respect to independent variable.
  For example:
  $x\frac{d^4y}{dx^4}+y\left(\frac{d^2y}{dx^2}\right)^3-2x^2y\frac{dy}{dx}+3=0$ is a differential equation.

  Sometimes, we may write $\frac{dy}{dx},\frac{d^2y}{dx^2},\frac{d^3y}{dx^3},\frac{d^4y}{dx^4}$ etc. as y',y'',y''',y'''' etc. respectively. Also, note that we cannot say that $\tan(y')+x=0$ is a differential equation.

- **Order and degree of a differential equation**
  - **Order of a differential equation** is defined as the order of the highest order derivative of dependent variable with respect to independent variable involved in the given differential equation.
    For example: The highest order derivative present in the differential equation $x^3y^5\,y''''-3x^2y''+xy\,y'-5=0$ is y''''. Therefore, the order of this differential equation is 4.
  - **Degree of a differential equation** is the highest power of the highest order derivative in it.
    For example: The degree of the differential equation $(y''')^2-2x(y'')^5-xy(y'')^2+y'=0$ is 2, since the highest power of the highest order derivative, y''', is 2.
  - If a differential equation is defined, then its order and degree are always positive integers.

- **General and particular solutions of a differential equation**
  - A function that satisfies the given differential equation is called a solution of a given differential equation.
  - The solution of a differential equation, which contains arbitrary constants, is called general solution (primitive) of the differential equation.
  - The solution of a differential equation, which is free from arbitrary constants i.e., the solution obtained from the general solution by giving particular values to arbitrary constants is called a particular solution of the differential equation.

- **Formation of differential equations**
  To form a differential equation from a given function, we differentiate the function successively as many times as the number of constants in the given function and then eliminate the arbitrary constants.

- **Methods of solving first order, first degree differential equations**
  - **Variable separable method**

This method is used to solve such an equation in which variables can be separated completely, i.e., terms containing *y* should remain with *dy* and terms containing *x* should remain with *dx*.

- **Homogeneous differential equation**

  A differential equation which can be expressed as $\frac{dy}{dx} = f(x, y)$ or $\frac{dx}{dy} = g(x, y)$, where $f(x, y)$ and $g(x, y)$ are homogenous functions of degree zero is called a homogenous differential equation. To solve such an equation, we have to substitute *y* = *vx* in the given differential equation and then solve it by variable separable method.

- **Linear differential equation**

  a) A differential equation which can be expressed in the form of $\frac{dy}{dx} + \text{P}y = \text{Q}$, where P and Q are constants or functions of *x* only, is called a first order linear differential equation.
  In this case, we find integrating factor (I.F.) by using the formula:
  $\text{I.F.} = e^{\int \text{P}dx}$
  Then, the solution of the differential equation is given by,
  $y\,(\text{I.F}) = \int (\text{Q} \times \text{I.F.})\,dx + \text{C}$

  b) A linear differential equation can also be of the form $\frac{dx}{dy} + \text{P}_1 x = \text{Q}_1$,
  where P_1 and Q_1 are constants or functions of y only.
  In this case, $\text{I.F.} = e^{\int \text{P}_1 dy}$
  And the solution of the differential equation is given by,
  $x\,(\text{I.F.}) = \int (\text{Q}_1 \times \text{I.F.})\,dy + \text{C}$

Chapter 10: Vector Algebra

❖ **Scalar**

The quantity which involves only one value, i.e. magnitude, is called a scalar quantity. For example: Time, mass, distance, energy, etc.

❖ **Vector**

The quantity which has both magnitude and a direction is called a vector quantity. For example: force, momentum, acceleration, etc.

❖ **Directed line**

A line with a direction is called a directed line. Let $\overrightarrow{\mathrm{AB}}$ be a directed line along direction B.

Here,

- The length of the line segment AB represents the magnitude of the above directed line. It is denoted by $\left|\overrightarrow{\mathrm{AB}}\right|$ or $|\vec{a}|$ or *a*.
- $\overrightarrow{\mathrm{AB}}$ represents the vector in the direction towards point B. Therefore, the vector represented in the above figure is $\overrightarrow{\mathrm{AB}}$. It can also be denoted by $\vec{a}$.
- The point A from where the vector $\overrightarrow{\mathrm{AB}}$ starts is called its initial point and the point B where the vector $\overrightarrow{\mathrm{AB}}$ ends is called its terminal point.

❖ **Position vector**

The position vector of a point P(x, y, z) with respect to the origin (0, 0, 0) is given by $\overrightarrow{\mathrm{OP}} = x\hat{i} + y\hat{j} + z\hat{k}$. This form of any vector is known as the component form.

Here,

- $\hat{i}$, $\hat{j}$, and $\hat{k}$ are called the unit vectors along the x-axis, y-axis, and z-axis respectively.
- x, y, and z are the scalar components (or rectangular components) along x-axis, y-axis, and z-axis respectively.
- $x\hat{i}, y\hat{j}, z\hat{k}$ are called vector components of $\overrightarrow{\mathrm{OP}}$ along the respective axes.
- The magnitude of $\overrightarrow{\mathrm{OP}}$ is given by $\left|\overrightarrow{\mathrm{OP}}\right| = \sqrt{x^2 + y^2 + z^2}$

❖ **Components and direction ratios**

- The scalar components of a vector are its direction ratios and represent its projections along the respective axis.
- The direction ratios of a vector $\vec{p} = a\hat{i} + b\hat{j} + c\hat{k}$ are a, b, and c.

  Here, a, b, and c respectively represent projections of $\vec{p}$ along x-axis, y-axis, and z-axis.

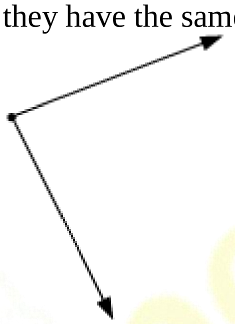
❖ **Direction cosines**

- The cosines of the angle made by the vector $\vec{r} = a\hat{i} + b\hat{j} + c\hat{k}$ with the positive directions of *x*, *y*, and *z* axes are its direction cosines. These are usually denoted by *l*, *m*, and *n*. Also, $l^2 + m^2 + n^2 = 1$
- The direction cosines (*l*, *m*, *n*) of a vector $a\hat{i} + b\hat{j} + c\hat{k}$ are
  $l = \frac{a}{r}, m = \frac{b}{r}, n = \frac{c}{r}$, where *r* = magnitude of the vector $a\hat{i} + b\hat{j} + c\hat{k}$

❖ **Types of vectors**

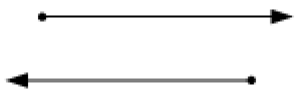
- **Zero vector:** A vector whose initial and terminal points coincide is called a zero vector (or null vector). It is denoted as $\vec{0}$. The vectors $\overrightarrow{AA}$, $\overrightarrow{BB}$ represent zero vectors.
- **Unit vector:** A vector whose magnitude is unity, i.e. 1 unit, is called a unit vector. The unit vector in the direction of any given vector $\vec{a}$ is denoted by $\hat{a}$ and it is calculated by $\hat{a} = \frac{1}{|\vec{a}|}\vec{a}$

  Note: if *l*, *m*, and *n* are direction cosines of a vector, then $l\hat{i} + m\hat{j} + n\hat{k}$ is the unit vector in the direction of that vector.
- **Co-initial vectors:** Two or more vectors are said to be co-initial vectors, if they have the same initial point.



- **Collinear vectors:** Two or more vectors are said to be collinear vectors, if they are parallel to a same line irrespective of their magnitude and direction.

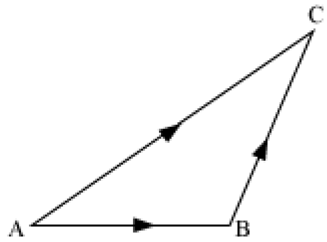
- **Equal vectors:** Two vectors $\vec{a}$ and $\vec{b}$ are said to equal, if they have same magnitude and direction regardless of the position of their initial points. They are written as $\vec{a} = \vec{b}$
- **Negative of a vector:** Two vectors are said to be negative of one another, if they have same magnitude, but their direction is opposite to one another.



For example, the negative of a vector $\overrightarrow{AB}$ is written as $\overrightarrow{BA} = -\overrightarrow{AB}$

❖ **Addition of vectors**

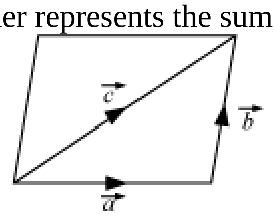
- **Triangle law of vector addition:** If two vectors are represented by two sides of a triangle in order, then the third closing side of the triangle in the opposite direction of the order represents the sum of the two vectors.



$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$

Note: The vector sum of the three sides of a triangle taken in order is $\vec{0}$

- **Parallelogram law of vector addition:** If two vectors are represented by two adjacent sides of a parallelogram in order, then the diagonal closing side of the triangle of the parallelogram in the opposite direction of the order represents the sum of two vectors.



$\vec{c} = \vec{a} + \vec{b}$

❖ **Properties of vector addition**

- Commutative property: $\vec{a} + \vec{b} = \vec{b} + \vec{a}$
- Associative property: $\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$
- Existence of additive identity: The vector $\vec{0}$ is additive identity of a vector $\vec{a}$, since $\vec{a} + \vec{0} = \vec{0} + \vec{a} = \vec{a}$
- Existence of additive inverse: The vector $-\vec{a}$ is called additive inverse of $\vec{a}$, since $\vec{a} + (-\vec{a}) = (-\vec{a}) + \vec{a} = 0$

❖ **Operations on vectors**

- The multiplication of vector $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ by any scalar λ is given by,
  $\lambda\vec{a} = (\lambda a_1)\hat{i} + (\lambda a_2)\hat{j} + (\lambda a_3)\hat{k}$
- The magnitude of the vector $\lambda\vec{a}$ is given by $|\lambda\vec{a}| = |\lambda||\vec{a}|$
- The sum of two vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ is given by,
  $\vec{a} + \vec{b} = (a_1 + b_1)\hat{i} + (a_2 + b_2)\hat{j} + (a_3 + b_3)\hat{k}$
- The difference of two vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ is given by $\vec{a} - \vec{b} = (a_1 - b_1)\hat{i} + (a_2 - b_2)\hat{j} + (a_3 - b_3)\hat{k}$

- **Equality of vectors**

  The vectors $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ are equal, if and only if $a_1 = b_1$, $a_2 = b_2$, and $a_3 = b_3$

- **Distributive law for vectors**

  Let $\overrightarrow{a_1}$ and $\overrightarrow{a_2}$ be two vectors, and k_1 and k_2 be any scalars, then the following are the distributive laws of addition and multiplication of a vector by a scalar:

  - $k_1\overrightarrow{a_1}+k_2\overrightarrow{a_1}=(k_1+k_2)\overrightarrow{a_1}$
  - $k_1\left(k_2\overrightarrow{a_1}\right)=\left(k_1k_2\right)\overrightarrow{a_1}$
  - $\overrightarrow{k_1}\left(\overrightarrow{a_1}+\overrightarrow{a_2}\right)=k_1\overrightarrow{a_1}+k_1\overrightarrow{a_2}$

- **Collinear vectors**

  - Two vectors $\vec{a}$ and $\vec{b}$ are collinear, if and only if there exists a non-zero scalar λ such that $\vec{b}=\lambda\vec{a}$
  - Two vectors $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ are collinear, if and only if $\frac{a_1}{b_1}=\frac{a_2}{b_2}=\frac{a_3}{b_3}$

- **Vector joining two points**

  The magnitude of the vector joining the two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is given by $\overline{P_1P_2}=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$

- **Section formula**

  The position vector of a point R dividing a line segment joining the points P and Q, whose position vectors are $\vec{a}$ and $\vec{b}$ respectively, in the ratio $m : n$

  - internally, is given by $\frac{n\vec{a}+m\vec{b}}{m+n}$
  - externally, is given by $\frac{m\vec{b}-n\vec{a}}{m-n}$

- **Scalar product of vectors**

  The scalar product of two non-zero vectors $\vec{a}$ and $\vec{b}$ is denoted by $\vec{a}\cdot\vec{b}$ and it is given by the formula $\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos\theta$, where θ is the angle between $\vec{a}$ and $\vec{b}$ such that $0 \le \theta \le \pi$

  If either $\vec{a}=0$ or $\vec{b}=0$, then in this case, θ is not defined and $\vec{a}\cdot\vec{b}=0$

  The following are the observations related to the scalar product of two vectors:

  - $\vec{a}\cdot\vec{b}$ is a real number.
  - The angle θ between vectors $\vec{a}$ and $\vec{b}$ is given by,

    $$\cos\theta=\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}\theta=\cos^{-1}\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}\right)$$

- $\vec{a}\cdot\vec{b}=0$, if and only if $\vec{a}\perp\vec{b}$
- If $\theta = 0$, then $\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|$
- If $\theta = \pi$, then $\vec{a}\cdot\vec{b}=-|\vec{a}||\vec{b}|$
- $\hat{i}\cdot\hat{i}=\hat{j}\cdot\hat{j}=\hat{k}\cdot\hat{k}=1, \hat{i}\cdot\hat{j}=\hat{j}\cdot\hat{k}=\hat{k}\cdot\hat{i}=0$
- If $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$, then $\vec{a}\cdot\vec{b}=a_1b_1+a_2b_2+a_3b_3$

❖ **Properties of scalar product**

- Commutative property: $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}$
- Distributivity of scalar product over addition: $\vec{a}\cdot(\vec{b}+\vec{c})=\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}$

❖ **Projection of a vector**

- If $\hat{p}$ is the unit vector along a line *l*, then the projection of a vector $\vec{a}$ on the line *l* is given by $\vec{a}.\hat{p}$.
- Projection of a vector $\vec{a}$ on other vector $\vec{b}$ is given by $\vec{a}\cdot\vec{b}$ or $\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$.

❖ **Vector product of vectors**

The vector product (or cross product) of two non-zero vectors $\vec{a}$ and $\vec{b}$ is denoted by $\vec{a}\times\vec{b}$ and is defined by $\vec{a}\times\vec{b}=|\vec{a}||\vec{b}|\sin\theta\hat{n}$, where θ is the angle between $\vec{a}$ and $\vec{b}$, $0 \leq \theta \leq \pi$, and $\hat{n}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$.

If $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ are two vectors, then their cross product $\vec{a}\times\vec{b}$, is defined by $\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$

The following are the observations made by the vector product of two vectors:

- $\vec{a}\times\vec{b}=\vec{0}$, if and only if $\vec{a}\parallel\vec{b}$
- $\hat{i}\times\hat{i}=\hat{j}\times\hat{j}=\hat{k}\times\hat{k}=0$

  $\hat{i}\times\hat{j}=\hat{k}, \hat{j}\times\hat{k}=\hat{i}, \hat{k}\times\hat{i}=\hat{j}$

  $\hat{j}\times\hat{i}=-\hat{k}, \hat{k}\times\hat{j}=-\hat{i}, \hat{i}\times\hat{k}=-\hat{j}$
- In terms of vector product, the angle θ between two vectors $\vec{a}$ and $\vec{b}$ is given by $\sin\theta=\dfrac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}$ or $\theta=\sin^{-1}\left(\dfrac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}\right)$
- If $\vec{a}$ and $\vec{b}$ represent the adjacent sides of a triangle, then its area is given as $\dfrac{1}{2}|\vec{a}\times\vec{b}|$.
- If $\vec{a}$ and $\vec{b}$ represent the adjacent sides of a parallelogram, then its area is given as $|\vec{a}\times\vec{b}|$.

- **Properties of vector product**
  - Not commutative: $\vec{a}\times\vec{b}\neq\vec{b}\times\vec{a}$

    However, $\vec{a}\times\vec{b}=-\vec{b}\times\vec{a}$
  - Distributivity of vector product over addition:

    $\vec{a}\times\left(\vec{b}+\vec{c}\right)=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}$

    $\lambda\left(\vec{a}\times\vec{b}\right)=\left(\lambda\vec{a}\right)\times\vec{b}=\vec{a}\times\left(\lambda\vec{b}\right)$

Chapter 11: Three Dimensional Geometry

- **Direction cosines (d.c.'s) of a line**
  - D.c.'s of a line are the cosines of angles made by the line with the positive direction of the coordinate axes.
  - If l, m, and n are the d.c.'s of a line, then $l^2 + m^2 + n^2 = 1$
  - D.c.'s of a line joining two points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) are $\frac{x_2 - x_1}{PQ}, \frac{y_2 - y_1}{PQ}, \frac{z_2 - z_1}{PQ}$, where PQ = $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

- **Direction ratios (d.r.'s) of a line**
  - D.r.'s of a line are the numbers which are proportional to the d.c.'s of the line.
  - D.r.'s of a line joining two points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) are given by $x_1 - x_2, y_1 - y_2, z_1 - z_2$ or $x_2 - x_1, y_2 - y_1, z_2 - z_1$.

- If a, b, and c are the d.r.'s of a line and l, m, and n are its d.c.'s, then
  - $\frac{l}{a} = \frac{m}{b} = \frac{n}{c}$
  - $l = \pm\frac{a}{\sqrt{a^2 + b^2 + c^2}}$, $m = \pm\frac{b}{\sqrt{a^2 + b^2 + c^2}}$, and $n = \pm\frac{c}{\sqrt{a^2 + b^2 + c^2}}$

- **Equation of a line through a given point and parallel to a given vector**
  - **Vector form**
    Equation of a line that passes through the given point whose position vector is $\vec{a}$ and which is parallel to a given vector $\vec{b}$ is
    $\vec{r} = \vec{a} + \lambda\vec{b}$, where λ is a constant.
  - **Cartesian form**
    - Equation of a line that passes through a point (x_1, y_1, z_1) having d.r.'s as a, b, c is given by $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$
    - Equation of a line that passes through a point (x_1, y_1, z_1) having d.c.'s as l, m, n is given by $\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$

- **Equation of a line passing through two given points**
  - **Vector form:** Equation of a line passing through two points whose position vectors are $\vec{a}$ and $\vec{b}$ is given by $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$, where $\lambda \in$ **R**
  - **Cartesian form:** Equation of a line that passes through two given points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by,
    $\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$

- **Skew lines and angle between them**

- Two lines in space are said to be **skew lines**, if they are neither parallel nor intersecting. They lie in different planes.
- Angle between two skew lines is the angle between two intersecting lines drawn from any point (preferably from the origin) parallel to each of the skew lines.

❖ **Angle between two non-skew lines**

- **Cartesian form**
  - If l_1, m_1, n_1, and l_2, m_2, n_2 are the d.c.'s of two lines and θ is the acute angle between them, then $\cos\theta = |l_1 l_2 + m_1 m_2 + n_1 n_2|$
  - If a_1, b_1, c_1 and a_2, b_2, c_2 are the d.r.'s of two lines and θ is the acute angle between them, then $\cos\theta = \left|\dfrac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}.\sqrt{a_2^2+b_2^2+c_2^2}}\right|$
- **Vector form**

  If θ is the acute angle between the lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_1$, then

  $$\cos\theta = \left|\frac{\vec{b}_1 \cdot \vec{b}_2}{|\vec{b}_1||\vec{b}_2|}\right|$$

❖ Two lines with d.r.'s a_1, b_1, c_1 and a_2, b_2, c_2 are

- perpendicular, if $a_1a_2 + b_1b_2 + c_1c_2 = 0$
- parallel, if $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$

❖ **Shortest distance between two skew lines:** The shortest distance is the line segment perpendicular to both the lines.

- **Vector form:** Distance between two skew lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$ is given by,

  $$d = \left|\frac{(\vec{b}_1 \times \vec{b}_2)\cdot(\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|}\right|$$

- **Cartesian form:** The shortest distance between two lines $\dfrac{x-x_1}{a_1} = \dfrac{y-y_1}{b_1} = \dfrac{z-z_1}{c_1}$ and $\dfrac{x-x_2}{a_2} = \dfrac{y-y_2}{b_2} = \dfrac{z-z_2}{c_2}$ is given by,

  $$d = \left|\frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2+(a_1b_2-a_2b_1)^2}}\right|$$

- The shortest distance between two parallel lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}$ is given by,

$$d = \left| \frac{\vec{b} \times (\vec{a}_2 - \vec{a}_1)}{|\vec{b}|} \right|$$

- **Equation of a plane in normal form**
  - **Vector form:** Equation of a plane which is at a distance of d from the origin, and the unit vector $\hat{n}$ normal to the plane through the origin is $\vec{r}.\hat{n} = d$, where $\vec{r}$ is the position vector of a point in the plane
  - **Cartesian form:** Equation of a plane which is at a distance d from the origin and the d.c.'s of the normal to the plane as l, m, n is $lx + my + nz = d$

- **Equation of a plane perpendicular to a given vector and passing through a given point**
  - **Vector form:** Equation of a plane through a point whose position vector is $\vec{a}$ and perpendicular to the vector $\vec{N}$ is $(\vec{r} - \vec{a}).\vec{N} = 0$, where $\vec{r}$ is the position vector of a point in the plane
  - **Cartesian form:** Equation of plane passing through the point (x_1, y_1, z_1) and perpendicular to a given line whose d.r.'s are A, B, C is
    $A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$

- **Equation of a plane passing through three non-collinear points**
  - **Cartesian form:** Equation of a plane passing through three non-collinear points (x_1, y_1, z_1), (x_2, y_2, z_2), and (x_3, y_3, z_3) is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

  - **Vector form:** Equation of a plane that contains three non-collinear points having position vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ is $(\vec{r} - \vec{a}) \cdot \left[(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a}) \right] = 0$, where $\vec{r}$ is the position vector of a point in the plane.

- **Intercept form of the equation of a plane**
  Equation of a plane having x, y, and z intercepts as a, b, and c respectively i.e., the equation of the plane that cuts the coordinate axes at $(a, 0, 0)$, $(0, b, 0)$, and $(0, 0, c)$ is given by,

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

- **Planes passing through the intersection of two planes**
  - **Vector form:** Equation of the plane passing through intersection of two planes $\vec{r}.\vec{n}_1 = d_1$ and $\vec{r}.\vec{n}_2 = d_2$ is given by,
    $\vec{r}.(\vec{n}_1 + \lambda\vec{n}_2) = d_1 + \lambda d_2$, where λ is a non-zero constant

- **Cartesian form:** Equation of a plane passing through the intersection of two planes $A_1x+B_1y+C_1z+D_1=0$ and $A_2x+B_2y+C_2z+D_2=0$ is given by,
  $(A_1x+B_1y+C_1z+D_1)+\lambda(A_2x+B_2y+C_2z+D_2)=0$, where λ is a non-zero constant

❖ **Co-planarity of two lines**

- **Vector form:** Two lines $\vec{r}=\vec{a}_1+\lambda\vec{b}_1$ and $\vec{r}=\vec{a}_2+\mu\vec{b}_2$ are co-planar, if
  $(\vec{a}_2-\vec{a}_1)\cdot(\vec{b}_1\times\vec{b}_2)=0$
- **Cartesian form:** Two lines $\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$ are co-planar, if
  $$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}=0$$

❖ **Angle between two planes:** The angle between two planes is defined as the angle between their normals.

- **Vector form:** If θ is the angle between the two planes $\vec{r}\cdot\vec{n}_1=d_1$ and $\vec{r}\cdot\vec{n}_2=d_2$, then $\cos\theta=\left|\frac{\vec{n}_1\cdot\vec{n}_2}{|\vec{n}_1||\vec{n}_2|}\right|$
  Note that if two planes are perpendicular to each other, then $\vec{n}_1.\vec{n}_2=0$; and if they are parallel to each other, then $\vec{n}_1$ is parallel to $\vec{n}_2$.
- **Cartesian form:** If θ is the angle between the two planes $A_1x+B_1y+C_1z+D_1=0$ and $A_2x+B_2y+C_2z+D_2=0$, then
  $$\cos\theta=\left|\frac{A_1A_2+B_1B_2+C_1C_2}{\sqrt{A_1^2+B_1^2+C_1^2}\sqrt{A_2^2+B_2^2+C_2^2}}\right|$$
  Note that if two planes are perpendicular to each other, then $A_1A_2+B_1B_2+C_1C_2=0$; and if they are parallel to each other, then $\frac{A_1}{A_2}=\frac{B_1}{B_2}=\frac{C_1}{C_2}$

❖ **Distance of a point from a plane**

- **Vector form:** The distance of a point, whose position vector is $\vec{a}$, from the plane $\vec{r}\cdot\hat{n}=d$ is $|d-\vec{a}\cdot\hat{n}|$.
  Note:

- If the equation of the plane is in the form of $\vec{r}\cdot\vec{N}=d$, where $\vec{N}$ is the normal to the plane, then the perpendicular distance is $\frac{\left|\vec{a}\cdot\vec{N}-d\right|}{\left|\vec{N}\right|}$.
- Length of the perpendicular from origin to the plane $\vec{r}\cdot\vec{N}=d$ is $\frac{|d|}{\left|\vec{N}\right|}$.

- **Cartesian form:** The distance from a point (x_1, y_1,z_1) to the plane Ax + By + Cz + D = 0 is $\left|\frac{Ax_1+By_1+Cz_1+D}{\sqrt{A^2+B^2+C^2}}\right|$.

❖ **Angle between a line and a plane:** The angle ϕ between a line $\vec{r}=\vec{a}+\lambda\vec{b}$ and the plane $\vec{r}\cdot\vec{n}=d$ is the complement of the angle between the line and the normal to the plane and is given by $\phi=\sin^{-1}\left|\frac{\vec{b}\cdot\vec{n}}{\left|\vec{b}\right|\left|\vec{n}\right|}\right|$

Chapter12: Linear Programming

- Problems which seek maximise (or minimise) of a linear function (say, of two variables x and y) subject to certain constraints as determined by a set of linear inequalities are called optimisation problems.

- A Linear Programming Problem (L.P.P.) is the one that is concerned with finding the optimal value (maximum or minimum value) of a linear function of several variables (called objective function), subject to the conditions that the variables are non-negative and satisfy a set of linear inequalities (called constraints). The variables are sometimes called the decision variables.
  For example: The following is an L.P.P.
  Maximize $Z = 10x + 12y$
  Subject to the following constraints:
  $5x + 3y \leq 30$ ... (1)
  $x + 2y \geq 2$ ... (2)
  $x \geq 0, y \geq 0$ ... (3)
  In this L.P.P, the objective function is $Z = 10x + 12y$
  The inequalities (1), (2), and (3) are called constraints.

- The common region determined by all the constraints including the non-negative constraints $x \geq 0$, $y \geq 0$ of a linear programming problem is called the **feasible region** (or solution region) for the problem. The region outside this feasible region is called **infeasible region**.

- Points within and on the boundary of the feasible region represent **feasible solutions** of the constraints. Any point outside the feasible region is an **infeasible solution**.

- Any point in the feasible region that gives the optimal value (maximum or minimum) of the objective function is called an **optimal solution**.

- **Fundamental theorems for solving linear programming problems**
  Theorem 1: Let R be the feasible region for a linear programming problem and let $Z = ax + by$ be the objective function. When Z has an optimal value, where the variables x and y are subject to constraints described by linear inequalities, this optimal value must occur at a corner point of the feasible region.
  Theorem 2: Let R be the feasible region for a linear programming problem, and let $Z = ax + by$ be the objective function. If R is bounded, then the objective function Z has both a maximum and a minimum value on R and each of these occurs at a corner point of R.

- If the feasible region is unbounded, then a maximum or a minimum may not exist. However, if it exists, then it must occur at a corner point of R.

- **Corner point method:** This method is used for solving a linear programming problem and it comprises of the following steps:

  Step 1) Find the feasible region of the L.P.P. and determine its corner points.

  Step 2) Evaluate the objective function $Z = ax + by$ at each corner point. Let M and m respectively be the largest and smallest values at these points.

  Step 3) If the feasible region is bounded, then M and m respectively are the maximum and minimum values of the objective function.

  If the feasible region is unbounded

  - If the open half plane determined by $ax + by > M$ has no point in common with the feasible region, then M is the maximum value of the objective function. Otherwise, the objective function has no maximum value.
  - If the open half plane determined by $ax + by < m$ has no point in common with the feasible region, then m is the minimum value of the objective function. Otherwise, the objective function has no minimum value.

- If two corner points of the feasible region are both optimal solutions of the same type, i.e. both produce the same maximum or minimum, then any point on the line segment joining these two points is also an optimal solution of the same type.

- A few important linear programming problems are: diet problems, manufacturing problems, transportation problems, and allocation problems.

Example 1:

A firm is engaged in breeding goats. The goats are fed on various products grown in the farm. They require certain nutrients, named A, B, and C. The goats are fed on two products P and Q. One unit of product P contains 12 units of A, 18 units of B, and 25 units of C, while one unit of product Q contains 24 units of A, 9 units of B, 25 units of C. The minimum requirement of A and B are 144 units and 108 units respectively whereas the maximum requirement of C is 250 units. Product P costs Rs 35 per unit whereas product Q costs Rs 45 per unit. Formulate this as a linear programming problem. How many units of each product may be taken to minimise the cost? Also find the minimum cost.

Solution:

Let x and y be the number of units taken from products P and Q respectively to minimise the cost. Mathematical formulation of the given L.P.P. is as follows:

Minimise $Z = 35x + 45y$

Subject to constraints

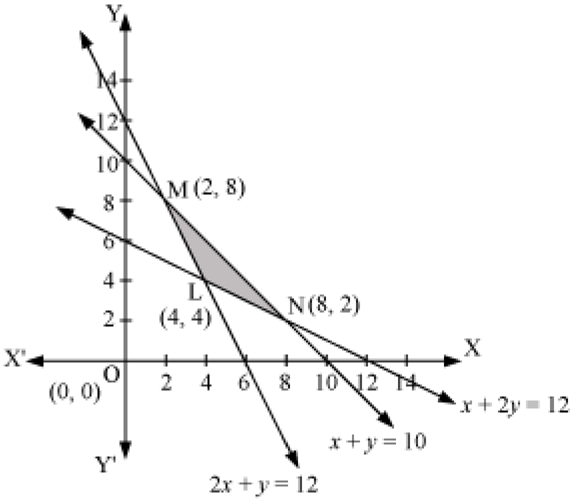
$12x + 24y \geq 144$ (constraints on A) $\Rightarrow x + 2y \geq 12$ ... (1)

$18x + 9y \geq 108$ (constraints on B) $\Rightarrow 2x + y \geq 12$ ... (2)

$25x + 25y \leq 250$ (constraints on C) $\Rightarrow x + y \leq 10$ ... (3)

$x \geq 0, y \geq 0$ ... (4)

The feasible region determined by the system of constraints is as follows:



The shaded region is the feasible region.

The corner points are L (4, 4), M (2, 8), and N (8, 2). The value of Z at these corner points are as follows:

Corner point	$Z = 35x + 45y$	
L (4, 4)	320	→ Minimum
M (2, 8)	430	
N (8, 2)	370	

It can be observed that the value Z is minimum at the corner point L (4, 4) and the minimum value is 320.

Therefore, 4 units of each of the products P and Q are taken to minimise the cost and the minimum cost is Rs 320.

Chapter 13: Probability

- **Conditional probability**

  If E and F are two events associated with the sample space of a random experiment, then the conditional probability of event E, given that F has already occurred, is denoted by P(E/F) and is given by the formula:

  $P(E/F) = \frac{P(E \cap F)}{P(F)}$, where $P(F) \neq 0$

- **Properties of conditional probability**

  If E and F are two events of a sample space S of an experiment, then the following are the properties of conditional probability:
  - $0 \leq P(E/F) \leq 1$
  - $P(F/F) = 1$
  - $P(S/F) = 1$
  - $P(E'/F) = 1 - P(E/F)$
  - If A and B are two events of a sample space S and F is an event of S such that $P(F) \neq 0$, then
    - $P((A \cup B)/F) = P(A/F) + P(B/F) - P((A \cap B)/F)$
    - $P((A \cup B)/F) = P(A/F) + P(B/F)$, if the events A and B are disjoint.

- **Multiplication theorem of probability**

  If E, F, and G are events of a sample space S of an experiment, then
  - $P(E \cap F) = P(E).\ P(F/E)$, if $P(E) \neq 0$
  - $P(E \cap F) = P(F).\ P(E/F)$, if $P(F) \neq 0$
  - $P(E \cap F \cap G) = P(E).\ P(F/E).\ P(G/(E \cap F)) = P(E).\ P(F/E).\ P(G/EF)$

- **Independent events**

  Two events E and F are said to be independent events, if the probability of occurrence of one of them is not affected by the occurrence of the other.
  - If E and F are two independent events, then
    - $P(F/E) = P(F)$, provided $P(E) \neq 0$
    - $P(E/F) = P(E)$, provided $P(F) \neq 0$
  - If three events A, B, and C are independent events, then $P(A \cap B \cap C) = P(A).\ P(B).\ P(C)$
  - If the events E and F are independent events, then
    - E' and F are independent
    - E' and F' are independent

- **Partition of a sample space**

  A set of events $E_1, E_2, \ldots E_n$ is said to represent a partition of the sample space S, if
  - $E_i \cap E_j = \phi,\ i \neq j,\ i, j = 1, 2, 3, \ldots n$
  - $E_1 \cup E_2 \cup \ldots \cup E_n = S$
  - $P(E_i) > 0,\ \forall\ i = 1, 2, 3, \ldots n$

- **Theorem of total probability**

Let $\{E_1, E_2, \ldots E_n\}$ be a partition of the sample space S, and suppose $P(E_i) > 0$, $\forall i = 1, 2, \ldots n$. Let A be any event associated with S, then

$P(A) = P(E_1).\ P(A/E_1) + P(E_2).\ P(A/E_2) + \ldots + P(E_n).\ P(A/E_n)$

$$= \sum_{j=1}^{n} P(E_j)P(A/E_j)$$

❖ Bayes' theorem

If $E_1, E_2, .. E_n$ are n non-empty events, which constitute a partition of sample space S, then

$$P(E_i / A) = \frac{P(E_i)P(A/E_i)}{\sum_{j=1}^{n} P(E_j)P(A/E_j)}, i = 1, 2, 3..., n$$

❖ Random variables and their probability distribution

- A random variable is a real-valued function whose domain is the sample space of a random experiment.
- The probability distribution of a random variable X is the system of numbers:

X:	x_1	x_2	…	x_n
$P(X)$:	p_1	p_2	…	p_n

Where, $p_i > 0 = \sum_{i=1}^{n} p_i = 1, i = 1, 2, ...n$

Here, the real numbers $x_1, x_2, \ldots, x_n$ are the possible values of the random variable X and p_i ($i = 1, 2, \ldots, n$) is the probability of the random variable X taking the value of x_i i.e., $P(X = x_i) = p_i$

❖ Mean/expectation of a random variable

Let X be a random variable whose possible values $x_1, x_2, x_3, \ldots x_n$ occur with probabilities $p_1, p_2, p_3, \ldots p_n$ respectively. The mean of X (denoted by μ) or the expectation of X (denoted by $E(X)$) is the number $\sum_{i=1}^{n} x_i p_i$.

That is, $E(X) = \mu = \sum_{i=1}^{n} x_i p_i = x_1 p_1 + x_2 p_2 + ... x_n p_n$

❖ Variance of a random variable

Let X be a random variable whose possible values $x_1, x_2, \ldots x_n$ occur with probabilities $p(x_1), p(x_2), \ldots p(x_n)$ respectively. Let $\mu = E(X)$ be the mean of X. The variance of X denoted by Var (X) or σ_x^2 is calculated by any of the following formulae:

- $\sigma_x{}^2 = \sum_{i=1}^{n} (x_i - \mu)^2 p(x_i)$
- $\sigma_x{}^2 = E(X - \mu)^2$

- $\sigma_x{}^2 = \sum_{i=1}^{n} x_i^2 p(x_i) - \left[\sum_{i=1}^{n} x_i p(x_i)\right]^2$
- $\sigma_x{}^2 = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$, where $[\mathrm{E}(X)]^2 = \sum_{i=1}^{n} x_i^2 p(x_i)$

It is advisable to students to use the fourth formula.

❖ **Standard deviation:** The non-negative number $\sigma_x = \sqrt{\mathrm{Var}\ (X)}$ is called the standard deviation of the random variable *X*.

$\sigma_x = \sqrt{\mathrm{E}(X^2) - [\mathrm{E}(X)]^2}$

❖ **Bernoulli trials:** Trials of a random experiment are called Bernoulli trials, if they satisfy the following conditions:

- There should be finite number of trials.
- The trials should be independent.
- Each trial has exactly two outcomes: success or failure.
- The probability of success remains the same in each trial.

❖ A binomial distribution with *n*-Bernoulli trials and probability of success in each trial as *p* is denoted by B(*n*, *p*).

❖ **Binomial distribution:** For binomial distribution B(*n*, *p*), the probability of *x* successes is denoted by P(*X* = *x*) or P(*X*) and is given by $\mathrm{P}(X = x) = {}^n\mathrm{C}_x q^{n-x} p^x, x = 0, 1, 2, \ldots n, q = 1 - p$

Here, P(*X*) is called the probability function of the binomial distribution.