Synopsis – Grade 12 Math Part I

Chapter 1: Relations and Functions

- **Relations**

  A relation R from a set A to a set B is a subset of $A \times B$ obtained by describing a relationship between the first element a and the second element b of the ordered pairs in $A \times B$. That is, $R \subseteq \{(a, b) \in A \times B, a \in A, b \in B\}$

  - **Domain of a relation**

    The domain of a relation R from set A to set B is the set of all first elements of the ordered pairs in R.

  - **Range of a relation**

    The range of a relation R from set A to set B is the set of all second elements of the ordered pairs in R. The whole set B is called the co-domain of R. Range $\subseteq$ Co-domain

- **Types of relations**

  - **Empty relations**

    A relation R in a set A is called an empty relation, if no element of A is related to any element of A. In this case, $R = \phi \subset A \times A$

    Example: Consider a relation R in set $A = \{3, 4, 5\}$ given by $R = \{(a, b)$: $a^b < 25$, where $a, b \in A\}$. It can be observed that no pair (a, b) satisfies this condition. Therefore, R is an empty relation.

  - **Universal relations**

    A relation R in a set A is called a universal relation, if each element of A is related to every element of A. In this case, $R = A \times A$

    Example: Consider a relation R in the set $A = \{1, 3, 5, 7, 9\}$ given by $R = \{(a, b)$: $a + b$ is an even number$\}$.

    Here, we may observe that all pairs (a, b) satisfy the condition R. Therefore, R is a universal relation.

  - **Note: Both the empty and the universal relation are called trivial relations.**

  - **Reflexive relations**

    A relation R in a set A is called reflexive, if $(a, a) \in R$ for every $a \in R$.

    Example: Consider a relation R in the set A, where $A = \{2, 3, 4\}$, given by $R = \{(a, b)$: $a^b = 4, 27$ or $256\}$. Here, we may observe that R = {(2, 2), (3, 3), and (4, 4)}. Since each element of R is related to itself (2 is related 2, 3 is related to 3, and 4 is related to 4), R is a reflexive relation.

  - **Symmetric relations**

    A relation R in a set A is called symmetric, if $(a_1, a_2) \in R \Rightarrow (a_2, a_1) \in R$, $\forall\, a_1, a_2 \in R$

    Example: Consider a relation R in the set A, where A is the set of natural numbers, given by $R = \{(a, b)$: $2 \leq ab < 20\}$. Here, it can be observed that $(b, a) \in R$ since $2 \leq ba < 20$ [since for natural numbers a and b, $ab = ba$]

    Therefore, the relation R symmetric.

- **Transitive relations**
  A relation R in a set A is called transitive, if $(a_1, a_2) \in R$ and $(a_2, a_3) \in R \Rightarrow (a_1, a_3) \in R$ for all $a_1, a_2, a_3 \in A$
  Example: Let us consider a relation R in the set of all subsets with respect to a universal set U given by $R = \{(A, B): A \text{ is a subset of } B\}$
  Now, if A, B, and C are three sets in R, such that $A \subset B$ and $B \subset C$, then we also have $A \subset C$. Therefore, the relation R is a symmetric relation.
- **Equivalence relations**
  A relation R in a set A is said to be an equivalence relation, if R is altogether reflexive, symmetric, and transitive.

❖ **Equivalence classes**

Given an arbitrary equivalence relation R in an arbitrary set X, R divides X into mutually disjoint subsets Ai called partitions or subdivisions of X satisfying:

- All elements of Ai are related to each other, for all i.
- No element of Ai is related to any element of Aj , $i \neq j$
- $\cup Aj = X$ and $Ai \cap Aj = \phi$, $i \neq j$

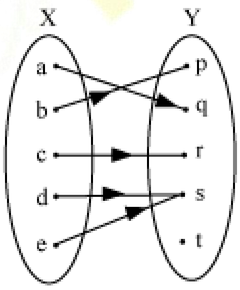
The subsets Ai are called equivalence classes.

❖ **Functions**

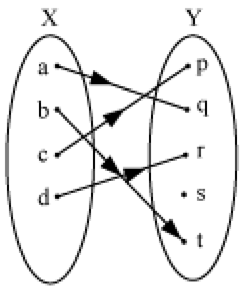
A function f from set X to Y is a specific type of relation in which every element x of X has one and only one image y in set Y. We write the function f as $f: X \to Y$, where $f(x) = y$

❖ **Types of functions**

- **One-one or injective and many-one functions**
  A function $f: X \to Y$ is said to be one-one or injective, if the image of distinct elements of X under f are distinct. In other words, if $x_1, x_2 \in X$ and $f(x_1) = f(x_2)$, then $x_1 = x_2$.
  If the function f is not one-one, then f is called a many-one function.
  The one-one and many-one functions can be illustrated by the following figures:

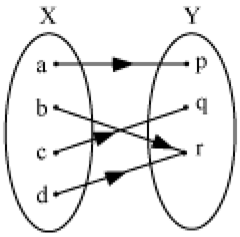


many-one

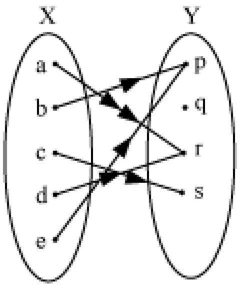


one-one

- **Onto (surjective) function**
  A function $f: X \to Y$ can be defined as an onto (surjective) function, if $\forall\, y \in Y$ such that there exists $x \in X$ such that $f(x) = y$
  The onto and many-one functions can be illustrated by the following figures:

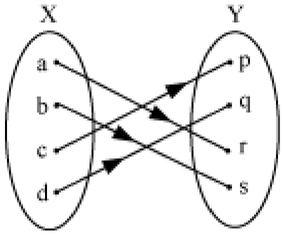


Onto



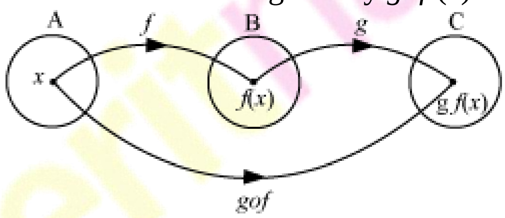
Not onto

- **One-one and onto (bijective) functions**
  A function $f: X \to Y$ is said to be bijective, if it is both one-one and onto. A bijective function can be illustrated by the following figure:



❖ **Composite function**

Let $f: A \to B$ and $g: B \to C$ be two functions. The composition of f and g, i.e. gof, is defined as a function from A to C given by $gof(x) = g(f(x)), \forall x \in A$



❖ **Inverse of function**

- A function $f: X \to Y$ is said to be **invertible**, if there exists a function $g: Y \to X$ such that $gof = I_X$ and $fog = I_Y$. In this case, g is called inverse of f and is written as $g = f^{-1}$
- A function f is invertible, if and only if f is bijective.

❖ **Binary operations**

A binary operation $*$ on a set A is a function $*$ from $A \times A$ to A

- An operation $*$ on a set A is commutative, if $a * b = b * a \ \forall a, b \in A$
- An operation $*$ on a set A is associative, if $(a * b) * c = a * (b * c) \ \forall a, b, c \in A$
- **Identity element**
  An element $e \in A$ is the identity element for binary operation $*: A \times A \to A$, if $a * e = a = e \times a \ \forall a \in A$
- **Inverse of the element**
  An element $a \in A$ is invertible for binary operation $*: A \times A \to A$, if there exists $b \in A$ such that $a * b = e = b * a$, where e is the identity for $*$. The element b is called inverse of a and is denoted by a^{-1}.

Chapter 2: Inverse Trigonometric Functions

❖ If $\sin y = x$, then $y = \sin^{-1}x$ (We read it as sine inverse x)
Here, $\sin^{-1}x$ is an inverse trigonometric function. Similarly, the other inverse trigonometric functions are as follows:

- If $\cos y = x$, then $y = \cos^{-1}x$
- If $\tan y = x$, then $y = \tan^{-1}x$
- If $\cot y = x$, then $y = \cot^{-1}x$
- If $\sec y = x$, then $y = \sec^{-1}x$
- If $\operatorname{cosec} y = x$, then $y = \operatorname{cosec}^{-1}x$

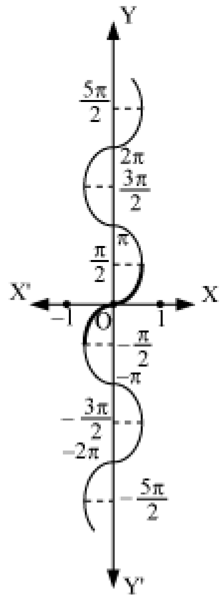
❖ The domains and ranges (principle value branches) of inverse trigonometric functions can be shown in a table as follows:

Function	Domain	Range (Principle value branches)
$y = \sin^{-1}x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$y = \cos^{-1}x$	$[-1, 1]$	$[0, \pi]$
$y = \tan^{-1}x$	**R**	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$y = \cot^{-1}x$	**R**	$(0, \pi)$
$y = \sec^{-1}x$	**R** – (–1, 1)	$[0, \pi] - \left\{\frac{\pi}{2}\right\}$
$y = \operatorname{cosec}^{-1}x$	**R** – (–1, 1)	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

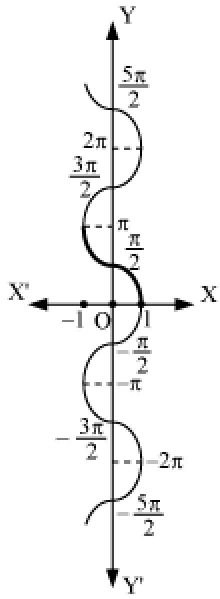
Note that $y = \tan^{-1}x$ does not mean that $y = (\tan x)^{-1}$. This argument also holds true for the other inverse trigonometric functions.

❖ The principal value of an inverse trigonometric function can be defined as the value of inverse trigonometric functions, which lies in its principal branch.

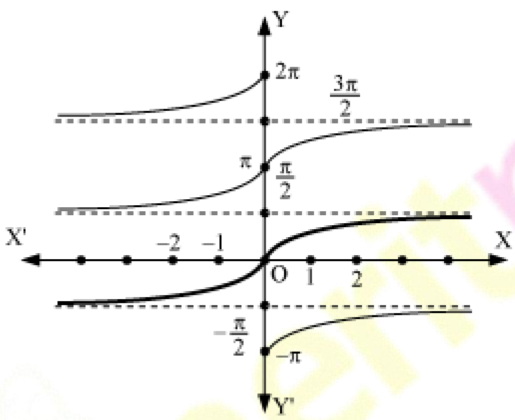
❖ **Graphs of the six inverse trigonometric functions**



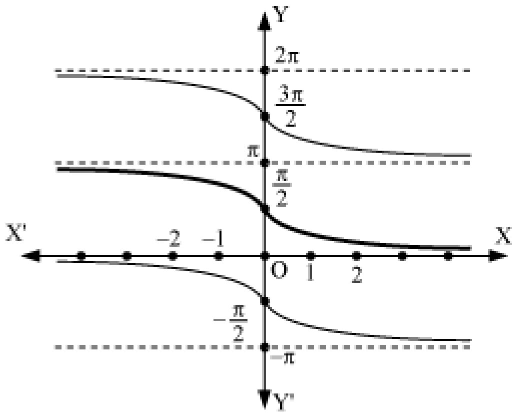
$y = \sin^{-1} x$



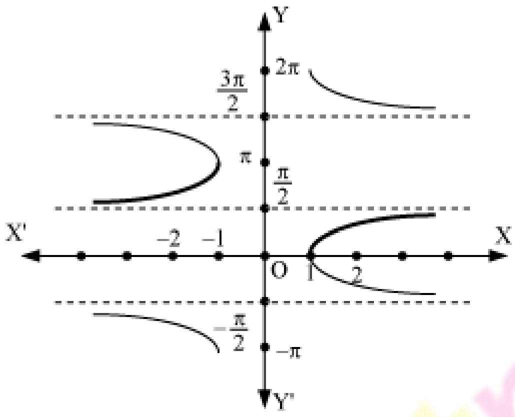
$y = \cos^{-1} x$



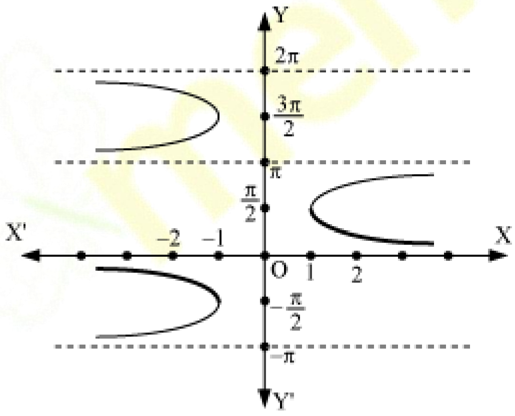
$y = \tan^{-1} x$



$y = \cot^{-1} x$



$y = \sec^{-1} x$



$y = \text{Cosec}^{-1} x$

❖ **Properties of inverse trigonometric functions**

- The relation $\sin y = x \Rightarrow y = \sin^{-1}x$ gives $\sin(\sin^{-1}x) = x$, where $-1 \le x \le 1$; and $\sin^{-1}(\sin x) = x$, where $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

  Similarly,
- $\cos(\cos^{-1}x) = x$, $-1 \le x \le 1$ and $\cos^{-1}(\cos x) = x$, $x \in [0, \pi]$
- $\tan(\tan^{-1}x) = x$, $x \in \mathbf{R}$ and $\tan^{-1}(\tan x) = x$, $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

- cosec (cosec^{-1}x) = x, $x \in$ **R** – (–1, 1) and cosec^{-1}(cosec x) = x, $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$
- sec (sec^{-1}x) = x, $x \in$ **R** – (–1, 1) and sec^{-1}(sec x) = x, $x \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$
- cot (cot^{-1}x) = x, $x \in$ **R** and cot^{-1}(cot x) = x, $x \in (0, \pi)$

❖ For suitable values of domains, we have

- $\sin^{-1}\left(\frac{1}{x}\right) = \text{cosec}^{-1}x$, $x \geq 1$ or $x \leq -1$
- $\cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1}x$, $x \geq 1$ or $x \leq -1$
- $\tan^{-1}\left(\frac{1}{x}\right) = \cot^{-1}x$, $x > 0$
- $\text{cosec}^{-1}\left(\frac{1}{x}\right) = \sin x$, $x \in$ **R** – (–1, 1)
- $\sec^{-1}\left(\frac{1}{x}\right) = \cos x$, $x \in$ **R** – (–1, 1)
- $\cot^{-1}\left(\frac{1}{x}\right) = \tan^{-1}x$, $x > 0$

❖ For suitable values of domains, we have

- $\sin^{-1}(-x) = -\sin^{-1}x$, $x \in [-1, 1]$
- $\cos^{-1}(-x) = \pi - \cos^{-1}x$, $x \in [-1, 1]$
- $\tan^{-1}(-x) = -\tan^{-1}x$, $x \in$ **R**
- $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1}x$, $|x| \geq 1$
- $\sec^{-1}(-x) = \pi - \sec^{-1}x$, $|x| \geq 1$
- $\cot^{-1}(-x) = \pi - \cot^{-1}x$, $x \in$ **R**

❖ For suitable values of domains, we have

- $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$, $x \in [-1, 1]$
- $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$, $x \in$ **R**
- $\sec^{-1}x + \text{cosec}^{-1}x = \frac{\pi}{2}$, $|x| \geq 1$

❖ For suitable values of domains, we have

- $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}, xy < 1$
- $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\frac{x-y}{1+xy}, xy > -1$

- For $x \in [-1, 1]$, we have $2\tan^{-1}x = \sin^{-1}\frac{2x}{1+x^2}$
- For $x \in (-1, 1)$, we have $2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}$
- For $x \geq 0$, we have $2\tan^{-1}x = \cos^{-1}\frac{1-x^2}{1+x^2}$

Chapter 3: Matrices

- A matrix is an ordered rectangular array of numbers or functions. The numbers or functions are called the elements or the entries of the matrix.

  For example: $\begin{bmatrix} -10 & \sin x & \log x \\ e^x & 2 & -9 \end{bmatrix}$ is a matrix having 6 elements. In this matrix, number of rows = 2 and number of columns = 3

- **Order of a matrix**

  A matrix having m rows and n columns is called a matrix of order $m \times n$. In such a matrix, there are mn numbers of elements. A matrix A of order $m \times n$ can be written as:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1j} & \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2j} & \ldots & a_{2n} \\ . \\ . \\ . \\ a_{i1} & a_{i2} & a_{i3} & \ldots & a_{ij} & \ldots & a_{in} \\ . \\ . \\ . \\ a_{m1} & a_{m2} & a_{m3} & \ldots & a_{mj} & \ldots & a_{mn} \end{bmatrix}_{m\times n}$$

  The above matrix A can be written as $\left[a_{ij}\right]_{m\times n}$, where $1 \le i \le m$ and $1 \le j \le n$, i, $j \in \mathbf{N}$

  For example: The order of the matrix $\begin{bmatrix} \sin x & \cos x \\ -1 & 1+\sin x \\ 0 & \cos x \end{bmatrix}$ is 3×2.

- **Types of matrices**
  - **Row matrix**

    A matrix A is said to be a row matrix, if it has only one row. In general, $A=\left[a_{ij}\right]_{1\times n}$ is a row matrix of order $1 \times n$.

    Example: $\begin{bmatrix} -9 & 6 & 5 & e & \sin x \end{bmatrix}$ is a row matrix of order 1×5.
  - **Column matrix**

    A matrix B is said to be a column matrix, if it has only one column. In general, $B=\left[b_{ij}\right]_{m\times 1}$ is a column matrix of order $m \times 1$.

Example: $B = \begin{bmatrix} -6 \\ 19 \\ 13 \end{bmatrix}$ is a column matrix of order 3 × 1.

- **Square matrix**
  A matrix C is said to be a square matrix, if the number of rows and columns of the matrix are equal. In general, $C = \left[b_{ij}\right]_{m \times n}$ is a square matrix, if $m = n$

  Example: $C = \begin{bmatrix} -1 & 9 \\ 5 & 1 \end{bmatrix}$ is a square matrix.

- **Diagonal matrix**
  A square matrix A is said to be a diagonal matrix, if all its non-diagonal elements are zero. In general, $A = \left[a_{ij}\right]_{m \times n}$ is a diagonal matrix, if $a_{ij} = 0$ for $i \neq j$

  Example: $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$ is a diagonal matrix.

- **Scalar matrix**
  A diagonal matrix is said to be a scalar matrix, if its diagonal elements are equal. In general, $A = \left[a_{ij}\right]_{m \times n}$ is a scalar matrix, if $a_{ij} = 0$ for $i \neq j$ and $a_{ij} = k$, for $i = j$, where k is constant.

  Example: $\begin{bmatrix} \sin 2x & 0 & 0 \\ 0 & \sin 2x & 0 \\ 0 & 0 & \sin 2x \end{bmatrix}$ is a scalar matrix.

- **Identity matrix**
  A square matrix in which all the diagonal elements are equal to 1 and the rest are all zero is called an identity matrix. It is denoted by I. In general, $I = \left[a_{ij}\right]_{m \times n}$ is an identity matrix, if $a_{ij} = 0$ for $i \neq j$ and $a_{ij} = 1$ for $i = j$

  Example: $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is an identity matrix.

- **Zero matrix**
  If all the elements of a matrix are zero, then it is called a zero matrix. It is denoted by O.

  Example: $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ is a zero matrix.

❖ **Equality of matrices**

Two matrices $A=\left[a_{ij}\right]$ and $B=\left[b_{ij}\right]$ are said to be equal matrices, if

- they are of same order
- $a_{ij} = b_{ij}$, for all possible values of i and j

❖ **Addition of matrices**

- Two matrices $A=\left[a_{ij}\right]$ and $B=\left[b_{ij}\right]$ can be added, if they are of the same order.
- The sum of two matrices A and B of same order $m \times n$ is defined as matrix $C=\left[c_{ij}\right]_{m\times n}$, where $c_{ij} = a_{ij} + b_{ij}$ for all possible values of i and j.

❖ **Multiplication of a matrix by a scalar**

The multiplication of a matrix A of order $m \times n$ by a scalar k is defined as

$kA = k\left[a_{ij}\right]_{m\times n} = \left[k(a_{ij})\right]_{m\times n}$

❖ **Difference of matrices**

- The **negative of a matrix** B is denoted by $-B$ and is defined as $(-1)B$.
- The difference of two matrices A and B is defined, if and only if they are of same order. The difference of the matrices A and B is defined as $A - B = A + (-1)B$

❖ **Properties of matrix addition**

If A, B, and C are three matrices of same order, then they follow the following properties related to addition:

- Commutative law: $A + B = B + A$
- Associative law: $A + (B + C) = (A + B) + C$
- Existence of additive identity: For every matrix A, there exists a matrix O such that $A + O = O + A = A$. In this case, O is called the additive identity for matrix addition.
- Existence of additive inverse: For every matrix A, there exists a matrix $(-A)$ such that $A + (-A) = (-A) + A = O$. In this case, $(-A)$ is called the additive inverse or the negative of A.

❖ **Properties of scalar multiplication of a matrix**

If A and B are matrices of same order and k and l are scalars, then

- $k(A + B) = kA + kB$
- $(k + l) A = kA + lA$

❖ **Multiplication of matrices**

The product of two matrices A and B is defined, if the number of columns of A is equal to the number of rows of B.

If $A=\left[a_{ij}\right]_{m\times n}$ and $B=\left[b_{ij}\right]_{n\times p}$ are two matrices, then their product is defined

as $AB = C = \left[c_{ik}\right]_{m\times p}$, where $c_{ik} = \sum_{j=1}^{n} a_{ij}b_{jk}$

Example: If $A=\begin{bmatrix}2 & -3 & 7\\ 0 & 1 & -9\end{bmatrix}$ and $B=\begin{bmatrix}-5 & 9\\ 7 & 2\\ 0 & 1\end{bmatrix}$, then find *AB*.

Solution:

$$AB=\begin{bmatrix}2 & -3 & 7\\ 0 & 1 & -9\end{bmatrix}\times\begin{bmatrix}-5 & 9\\ 7 & 2\\ 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}2\times(-5)+(-3)\times 7+7\times 0 & 2\times 9+(-3)\times 2+7\times 1\\ 0\times(-5)+1\times 7+(-9)\times 0 & 0\times 9+1\times 2+(-9)\times 1\end{bmatrix}$$

$$=\begin{bmatrix}-31 & 19\\ 7 & -7\end{bmatrix}$$

❖ **Properties of multiplication of matrices**

If *A*, *B*, and *C* are any three matrices, then they follow the following properties related to multiplication:

- Associative law: (*AB*) *C* = *A* (*BC*)
- Distribution law: *A* (*B* + *C*) = *AB* + *AC* and (*A* + *B*) *C* = *AC* + *BC*, if both sides of equality are defined.
- Existence of multiplicative identity: For every square matrix *A*, there exists an identity matrix *I* of same order such that *IA* = *AI* = *A*. In this case, *I* is called the multiplicative identity.
- Multiplication of two matrices is not commutative. There are many cases where the product *AB* of two matrices *A* and *B* is defined, but the product *BA* need not be defined.

❖ **Transpose of a matrix**

If *A* is an $m \times n$ matrix, then the matrix obtained by interchanging the rows and columns of *A* is called the transpose of matrix *A*. The transpose of *A* is denoted by A' or A^T. In other words, if $A=\left[a_{ij}\right]_{m\times n}$, then $A'=\left[a_{ij}\right]_{n\times m}$

Example: The transpose of the matrix $\begin{bmatrix}2 & 8 & -3\\ 1 & 11 & 9\end{bmatrix}$ is $\begin{bmatrix}2 & 1\\ 8 & 11\\ -3 & 9\end{bmatrix}$.

❖ **Properties of transpose of matrices**

- $(A')' = A$
- $(kA)' = kA'$, where k is a constant
- $(A+B)' = A' + B'$
- $(AB)' = B'A'$

❖ **Symmetric and skew symmetric matrices**

- If *A* is square matrix such that $A' = A$, then *A* is called a symmetric matrix.

- If A is a square matrix such that $A' = -A$, then A is called a skew symmetric matrix.
- For any square matrix A with entries as real numbers, $A + A'$ is a symmetric matrix and $A - A'$ is a skew symmetric matrix.
- Every square matrix can be expressed as the sum of a symmetric matrix and a skew symmetric matrix. In other words, if A is any matrix, then A can be expressed as $P + Q$, where $P = \frac{1}{2}(A + A')$ and $Q = \frac{1}{2}(A - A')$

❖ **Elementary operations or transformations on a matrix**

The various elementary operations or transformations on a matrix are as follows:

- $R_i \leftrightarrow R_j$ or $C_i \leftrightarrow C_j$
- $R_i \leftrightarrow kR_i$ or $C_i \leftrightarrow kC_j$
- $R_i \leftrightarrow R_i + kR_j$ or $C_i \leftrightarrow C_i + kC_j$

Example: By applying $R_1 \to R_1 - 7R_3$ to the matrix $\begin{bmatrix} -9 & 5 & 8 \\ 5 & 6 & 11 \\ 2 & -1 & 0 \end{bmatrix}$, we obtain $\begin{bmatrix} -23 & 12 & 8 \\ 5 & 6 & 11 \\ 2 & -1 & 0 \end{bmatrix}$.

❖ **Inverse of a matrix**

- If A and B are the square matrices of same order such that $AB = BA = I$, then B is called the inverse of A and A is called the inverse of B. i.e., $A^{-1} = B$ and $B^{-1} = A$
- If A and B are invertible matrices of the same order, then $(AB)^{-1} = B^{-1} A^{-1}$
- Inverse of a square matrix, if it exists, is unique.
- If the inverse of a matrix exists, then it can be calculated either by using elementary row operations or by using elementary column operations.

Chapter 4: Determinants

- Determinant of a square matrix A is denoted by $|A|$ or det (A).

- Determinant of a matrix $A=[a]_{1\times1}$ is $|A|=|a|=a$

- Determinant of a matrix $A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is given by,

  $$|A|=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}=a_{11}a_{22}-a_{12}a_{21}$$

- **Expansion of a determinant of a matrix**

  Determinant of a matrix $A=\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$ is given by (expanding along R_1):

  $$|A|=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}=(-1)^{1+1}a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix}+(-1)^{1+2}a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}+(-1)^{1+3}a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

  $$=a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix}-a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}+a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

  Similarly, we can find the determinant of A by expanding along any other row or along any column.

- If $A = kB$, where A and B are square matrices of order n, then $|A|=k^n|B|$, where n = 1, 2, 3

- **Properties of determinants**
  - If the rows and the columns of a square matrix are interchanged, then the value of the determinant remains unchanged.

    This property is same as saying, if A is a square matrix, then $|\mathrm{A}|=|\mathrm{A}'|$
  - If we interchange any two rows (or columns), then sign of determinant changes.
  - If any two rows or any two columns of a determinant are identical or proportional, then the value of the determinant is zero.
  - If each element of a row or a column of determinant is multiplied by a constant α, then its determinant value gets multiplied by α.
  - If $A=\left[a_{ij}\right]_{3\times3}$, then $|kA|=k^3|A|$

- If elements of a row or a column in a determinant can be expressed as sum of two (or more) elements, then the given determinant can be expressed as sum of two (or more) determinants.

  Example:

$$\begin{vmatrix} a_1+d_1 & b_1 & c_1 \\ a_2+d_2 & b_2 & c_2 \\ a_3+d_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

- If the equi-multiples of corresponding elements of another row or column are added to each element of any row or column of a determinant, then the value of the determinant remains unchanged.

$$\begin{vmatrix} a_1+\lambda a_3 & b_1+\lambda b_3 & c_1+\lambda c_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

❖ **Area of a triangle**

Area of a triangle with vertices (x_1, y_1), (x_2, y_2), and (x_3, y_3) is given by,

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Since area is always positive, we take the absolute value of the above determinant.

❖ **Minors and cofactors**

- Minor of an element a_{ij} (denoted by M_{ij}) of the determinant of matrix A is the determinant obtained by deleting its i^{th} row and j^{th} column.

  Example:

$$\text{If } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \text{ then } M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

- Co-factor of an element a_{ij} is defined by $A_{ij} = (-1)^{i+j} M_{ij}$, where M_{ij} is the minor of a_{ij}.

  Example:

$$\text{If } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \text{ then } A_{23} = (-1)^{2+3} M_{23} = -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

- If element of a row (or column) are multiplied with co-factors of any other row (or column), then their sum is zero.

  For example: If A is square matrix of order 3, then

$$a_{21} A_{11} + a_{22} A_{12} + a_{23} A_{23} = 0$$

- If elements of one row or column of a determinant is multiplied with its corresponding co-factors, then their sum is equal to the value of determinant.

  Example:

  If A is a square matrix of order 3, then $a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33} = |A|$

❖ Adjoint and inverse of a matrix

- If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then $\text{A}dj\ A = \begin{bmatrix} \text{A}_{11} & \text{A}_{21} & \text{A}_{31} \\ \text{A}_{12} & \text{A}_{22} & \text{A}_{32} \\ \text{A}_{13} & \text{A}_{23} & \text{A}_{33} \end{bmatrix}$,where A_{ij} is the co-factor of a_{ij} .
- If A is a square matrix, then $A\ (AdjA) = (AdjA)\ A = |A|\ \text{I}$
- A square matrix A is said to be **singular**, if $|A| = 0$
- A square matrix A is said to be **non-singular**, if $|A| \neq 0$
- If A and B are non-singular matrices of same order, then AB and BA are also non-singular matrices of same order.
- If A and B are square matrices of same order, then $|AB| = |A||B|$
- If A is a non-singular matrix of order n, then $|(AdjA)| = |A|^{n-1}$
- A square matrix A is invertible, if and only if A is non-singular and inverse of A is given by the formula:

  $$A^{-1} = \frac{1}{|A|}(AdjA)$$

❖ Application of determinants and matrices

The system of following linear equations

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

can be written as $AX = B$, where

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

- **Consistent system:** A system of linear equations is said to be consistent, if its solution (one or more) exists.
- **Inconsistent system:** A system of linear equations is said to be inconsistent, if its solution does not exist.
- Unique solution of equation $AX = B$ is given by $X = A^{-1}\ B$, where $|A| \neq 0$
- For a square matrix A in equation $AX = B$, if

  a) $|A| \neq 0$, then there exists unique solution

b) $|A| = 0$ and (adj$A$) $B \neq 0$, then no solution exists

c) $|A| = 0$ and (adj$A$) $B = 0$, then the system may or may not be consistent

Chapter 5: Continuity and Differentiability

❖ Continuity

- Suppose f is a real function on a subset of the real numbers and c be a point in the domain of f. Then, f is continuous at c, if $\lim_{x \to c} f(x) = f(c)$

  More elaborately, we can say that f is continuous at c, if

  $\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = f(c)$
- If f is not continuous at c, then we say that f is discontinuous at c and c is called the point of discontinuity.
- A real function f is said to be continuous, if it is continuous at every point in the domain of f.

❖ Algebra of continuous functions

- If f and g are two continuous real functions, then
- $(f + g)(x)$, $(f - g)(x)$, $f(x).g(x)$ are continuous
- $\dfrac{f(x)}{g(x)}$ is continuous, if $g(x) \neq 0$
- If f and g are two continuous functions, then fog is also continuous.

❖ Differentiability

- Suppose f is a real function and c is a point in its domain. Then, the derivative of f at c is defined by,

  $$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$
- Derivative of a function $f(x)$, denoted by $\frac{d}{dx}(f(x))$ or $f'(x)$, is defined by

  $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Example: Find derivative of sin $2x$.

Solution:

Let $f(x) = \sin 2x$

$$\therefore f'(x) = \lim_{h \to 0} \frac{\sin 2(x+h) - \sin 2x}{h}$$

$$= \lim_{h \to 0} \frac{2\cos(2x+h) \cdot \sin h}{h}$$

$$= 2 \lim_{h \to 0} \cos(2x+h) \cdot \lim_{h \to 0} \frac{\sin h}{h}$$

$$= 2 \times \cos 2x \times 1$$

$$= 2\cos 2x$$

❖ Algebra of derivatives

- $(f+g)' = f' + g'$
- $(f-g)' = f' - g'$
- $(fg)' = f'g'$
- $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$, where $g \neq 0$

❖ Every differentiable function is continuous, but the converse is not true.

❖ **Derivative of a composite function**
Chain rule: This rule is used to find the derivative of a composite function. If $f = v \text{ o } u$, then $t = u(x)$; and if both $\frac{dt}{dx}$ and $\frac{dv}{dt}$ exist, then $\frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx}$
Similarly, if $f = (w \text{ o } u) \text{ o } v$, and if $t = v(x)$, $s = u(t)$, then $\frac{df}{dx} = \frac{d(w o u)}{dt} \cdot \frac{dt}{dx} = \frac{dw}{ds} \cdot \frac{ds}{dt} \cdot \frac{dt}{dx}$

❖ **Derivatives of some useful functions**
- $\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}\left(\cos^{-1} x\right) = \frac{-1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$
- $\frac{d}{dx}\left(\cot^{-1} x\right) = \frac{-1}{1+x^2}$
- $\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{x\sqrt{x^2-1}}$
- $\frac{d}{dx}\left(\text{cosec}^{-1} x\right) = \frac{-1}{x\sqrt{x^2-1}}$
- $\frac{d}{dx}\left(\log x\right) = \frac{1}{x}$
- $\frac{d}{dx}\left(e^x\right) = e^x$
- $\frac{d}{dx}\left(e^{ax}\right) = ae^{ax}$

❖ **Properties of logarithmic functions**
- $\log_a xy = \log_a x + \log_a y$
- $\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$
- $\log_a x^n = n \log_a x$

- **Logarithmic differentiation**

  Derivative of a function $f(x)=[ux]^{v(x)}$ can be calculated by taking logarithm on both sides, i.e., $\log f(x)=v(x)\log[u(x)]$, and then differentiating both sides with respect to *x*.

- **Derivatives of functions in parametric forms**

  If the variables *x* and *y* are expressed in form of *x* = *f* (*t*) and *y* = *g* (*t*), then they are said to be in parametric form. In this case, $\frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{dx}=\frac{g'(t)}{f'(t)}$, provided $f'(t)\neq 0$

- **Second order derivative**

  If *y* = *f* (*x*), then $\frac{dy}{dx}=f'(x)$ and $\frac{d^2y}{dx^2}$ or $f''(x)=\frac{d}{dx}\left(\frac{dy}{dx}\right)$

  Here, $f''(x)$ or $\frac{d^2y}{dx^2}$ is called the second order derivative of *y* with respect to *x*.

- **Rolle's theorem**

  If *f*: [*a*, *b*] → **R** is continuous on [*a*, *b*] and differentiable on (*a*, *b*) such that *f* (*a*) = *f* (*b*), where *a* and *b* are some real numbers, then there exists some *c* ∈ (*a*, *b*) such that $f'(c)=0$

- **Mean value theorem**

  If *f*: [*a*, *b*] → **R** is continuous on [*a*, *b*] and differentiable on (*a*, *b*), then there exists some *c* ∈ (*a*, *b*) such that $f'(c)=\frac{f(b)-f(a)}{b-a}$

Chapter 6: Application of Derivatives

❖ Rate of change of quantities

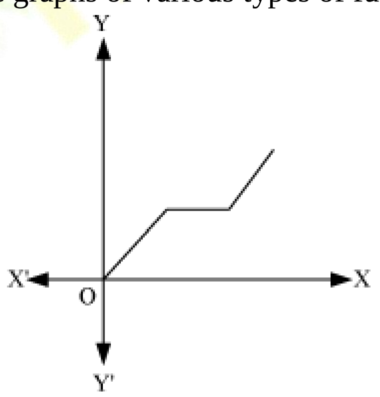
- For a quantity y varying with another quantity x, satisfying the rule $y = f(x)$, the rate of change of y with respect to x is given by $\frac{dy}{dx}$ or $f'(x)$.
- The rate of change of y with respect to the point $x = x_0$ is given by $\left.\frac{dy}{dx}\right]_{x=x_0}$ or $f'(x_0)$.
- If the variables x and y are expressed in form of $x = f(t)$ and $y = g(t)$, then the rate of change of y with respect to x is given by $\frac{dy}{dx} = \frac{g'(t)}{f'(t)}$, provided $f'(t) \neq 0$

❖ Increasing and decreasing functions

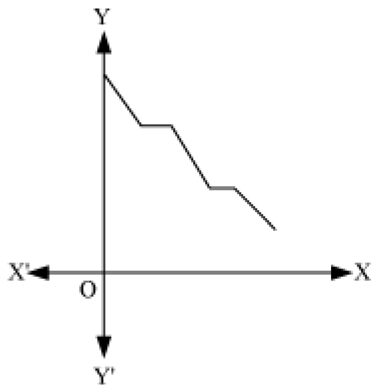
- A function $f: (a, b) \to \mathbf{R}$ is said to be
  i. increasing on (a, b), if $x_1 < x_2$ in $(a, b) \Rightarrow f(x_1) \leq f(x_2) \forall x_1, x_2 \in (a, b)$
  ii. decreasing on (a, b), if $x_1 < x_2$ in $(a, b) \Rightarrow f(x_1) \geq f(x_2) \forall x_1, x_2 \in (a, b)$

OR

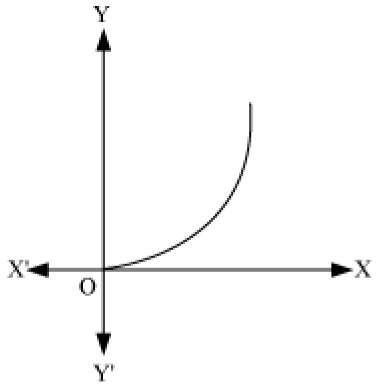
- If a function f is continuous on $[a, b]$ and differentiable on (a, b), then
  i. f is increasing in $[a, b]$, if $f'(x) > 0$ for each $x \in (a, b)$
  ii. f is decreasing in $[a, b]$, if $f'(x) < 0$ for each $x \in (a, b)$
  iii. f is constant function in $[a, b]$, if $f'(x) = 0$ for each $x \in (a, b)$
- A function $f: (a, b) \to \mathbf{R}$ is said to be
  i. strictly increasing on (a, b), if $x_1 < x_2$ in $(a, b) \Rightarrow f(x_1) < f(x_2) \forall x_1, x_2 \in (a, b)$
  ii. strictly decreasing on (a, b), if $x_1 < x_2$ in $(a, b) \Rightarrow f(x_1) > f(x_2) \forall x_1, x_2 \in (a, b)$
- The graphs of various types of functions can be shown as follows:



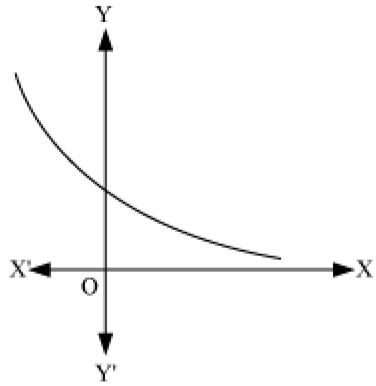
Increasing Function



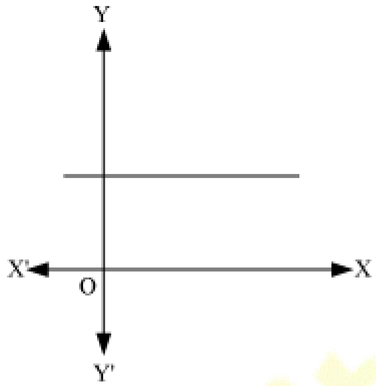
Decreasing Function



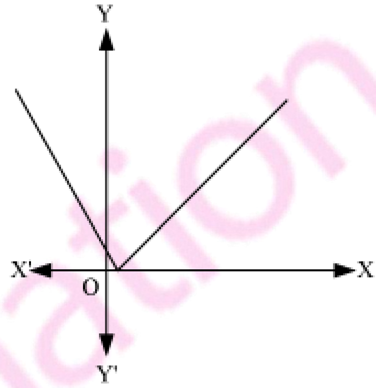
Strictly increasing function



Strictly decreasing function



Constant function



Neither increasing nor decreasing function

❖ Tangents and normals

- For the curve $y = f(x)$, the slope of tangent at the point (x_0, y_0) is given by $\left.\frac{dy}{dx}\right]_{(x_0, y_0)}$ or $f'(x_0)$.
- For the curve $y = f(x)$, the slope of normal at the point (x_0, y_0) is given by $\frac{-1}{\left.\frac{dy}{dx}\right]_{(x_0, y_0)}}$ or $\frac{-1}{f'(x_0)}$.
- The equation of tangent to the curve $y = f(x)$ at the point (x_0, y_0) is given by, $y - y_0 = f'(x_0) \times (x - x_0)$
- If $f'(x_0)$ does not exist, then the tangent to the curve $y = f(x)$ at the point (x_0, y_0) is parallel to the y-axis and its equation is given by $x = x_0$
- The equation of normal to the curve $y = f(x)$ at the point (x_0, y_0) is given by, $y - y_0 = \frac{-1}{f'(x_0)}(x - x_0)$
- If $f'(x_0)$ does not exist, then the normal to the curve $y = f(x)$ at the point (x_0, y_0) is parallel to the x-axis and its equation is given by $y = y_0$
- If $f'(x_0) = 0$, then the respective equations of the tangent and normal to the curve $y = f(x)$ at the point (x_0, y_0) are $y = y_0$ and $x = x_0$

❖ Approximations

Let $y = f(x)$ and let Δx be a small increment in x and Δy be the increment in y corresponding to the increment in x i.e., $\Delta y = f(x + \Delta x) - f(x)$

Then, $dy = f'(x)dx$ or $dy = \left(\frac{dy}{dx}\right)\Delta x$ is a good approximation of Δy, when $dx = \Delta x$ is relatively small and we denote it by $dy \approx \Delta y$

❖ Maxima and minima

Let a function f be defined on an interval I. Then, f is said to have

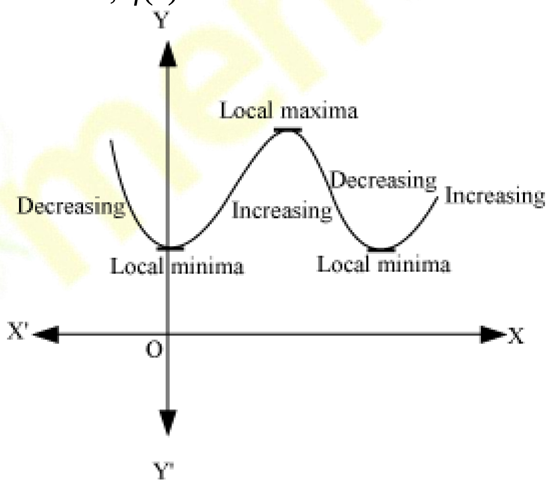
- maximum value in I, if there exists $c \in$ I such that $f(c) > f(x)$, $\forall\ x \in$ I [In this case, c is called the point of maxima]
- minimum value in I, if there exists $c \in$ I such that $f(c) < f(x)$, $\forall\ x \in$ I [In this case, c is called the point of minima]
- an extreme value in I, if there exists $c \in$ I such that c is either point of maxima or point of minima [In this case, c is called an extreme point]

  Note: Every continuous function on a closed interval has a maximum and a minimum value.

❖ Local maxima and local minima

Let f be a real-valued function and c be an interior point in the domain of f. Then c is called a point of

- local maxima, if there exists $h > 0$ such that $f(c) > f(x)$, $\forall\ x \in (c - h, c + h)$ [In this case, $f(c)$ is called the local maximum value of f]
- local minima, if there exists $h > 0$ such that $f(c) < f(x)$, $\forall\ x \in (c - h, c + h)$ [In this case, $f(c)$ is called the local maximum value of f]



❖ Critical point:

A point c in the domain of a function f at which either $f'(c) = 0$ or f is not differentiable is called a critical point of f.

❖ First derivative test

Let f be a function defined on an open interval I. Let f be continuous at a critical point c in I. Then:

- If $f'(x)$ changes sign from positive to negative as x increases through c, i.e. if $f'(x) > 0$ at every point sufficiently close to and to the left of c, and

$f'(x) < 0$ at every point sufficiently close to and to the right of *c*, then *c* is a point of local maxima.

- If $f'(x)$ changes sign from negative to positive as *x* increases through *c*, i.e. if $f'(x) < 0$ at every point sufficiently close to and to the left of *c*, and $f'(x) > 0$ at every point sufficiently close to and to the right of *c*, then *c* is a point of local minima.
- If $f'(x)$ does not change sign as *x* increases through *c*, then *c* is neither a point of local maxima nor a point of local minima. Such a point *c* is called point of inflection.

❖ **Second derivative test**

Let *f* be a function defined on an open interval I and $c \in$ I. Let *f* be twice differentiable at *c* and $f'(c) = 0$. Then:

- If $f''(c) < 0$, then *c* is a point of local maxima. In this situation, *f*(*c*) is local maximum value of *f*.
- If $f''(c) > 0$, then *c* is a point of local minima. In this situation, *f*(*c*) is local minimum value of *f*.
- If $f''(c) = 0$, then the test fails. In this situation, we follow first derivative test and find whether *c* is a point of maxima or minima or a point of inflection.

❖ **Absolute maximum value or absolute minimum value**

- Let *f* be a differentiable and continuous function on a closed interval, then *f* always attains its maximum and minimum value in the interval I, which are respectively known as the absolute maximum and absolute minimum value of *f*. Also, *f* attains these values at least once each in [*a*, *b*].
- Let *f* be a differentiable function on a closed interval I and *c* be any interior point of I such that $f'(c) = 0$, then *f* attains its absolute maximum value and its absolute minimum value at *c*.
- To find the absolute maximum value or/and absolute minimum value, we follow the steps listed below:

  Step 1: Find all critical points *f* in the interval.

  Step 2: Take the end point of interval.

  Step 3: Calculate the values of *f* at the points found in step 1 and step 2.

  Step 4: Identify the maximum and minimum values of *f* out of values calculated in step 3.

  The maximum value will be the absolute maximum (greatest) value of *f* and the minimum value will be the absolute minimum (least) value of *f*.