CRPF PUBLIC SCHOOL, ROHINI

THIRD Intra School Mathematics Olympiad 2012

CLASS XII

SOLUTIONS

SECTION – A

ANS 1. (B)

Multiplying the inequation by the product of $3\times5\times7=105$ results in

$$\frac{x-2}{5}<\frac{8}{3}<\frac{x+6}{7}$$

$$\frac{(105)(x-2)}{5}<\frac{(105)(8)}{3}<\frac{(105)(x+6)}{7}$$

$$21(x-2)<280<15(x+6)$$

In order to solve the inequality then,

$21x-42<280$	and	$15x+90>280$	and	$21x-42<15x+90$
$21x<322$	and	$15x>190$	and	$6x<132$
$x<15\frac{1}{3}$	and	$x>12\frac{2}{3}$	and	$x<22$

Therefore, the only integers that satisfy all three conditions are 13, 14, and 15.

ANS 2.

Solution 1: Using the sum and product of the roots of $ax^2+bx+c=0$. Let m and n be the roots. Then $m+n=\frac{-b}{a}=\frac{-(-7)}{18}=\frac{7}{18}$ and $m\times n=\frac{c}{a}=\frac{-1}{18}$.

We are given $\left(m+\frac{1}{3}\right)\left(n+\frac{1}{3}\right)=m\times n+\frac{1}{3}(m+n)+\frac{1}{9}$.

Substituting the known values:

$$\left(\frac{-1}{18}\right)+\frac{1}{3}\left(\frac{7}{18}\right)+\frac{1}{9}=\frac{5}{27}.$$

Solution 2: Using the Quadratic Formula,

$$x=\frac{-(-7)\pm\sqrt{7^2-4(18)(-1)}}{2(18)}$$

$$x=\frac{7\pm11}{36}$$

$$x=-\frac{1}{9}\text{ or }x=\frac{1}{2}$$

Since $m<0$ and $n>0$, let $m=-\frac{1}{9}$ and $n=\frac{1}{2}$. Thus

$$\left(m+\frac{1}{3}\right)\left(n+\frac{1}{3}\right)$$

$$=\left(-\frac{1}{9}+\frac{1}{3}\right)\left(\frac{1}{2}+\frac{1}{3}\right)$$

$$=\frac{2}{9}\times\frac{5}{6}$$

$$=\frac{5}{27}$$

ANS 3.

For the father to be 4 times as old as his son, the sum of their ages has to be a multiple of 5. Therefore, you have to add 2 to 33. Both have aged 1 year.
[Therefore, at present the father is 27 and the son is 6. In 1 year the father will be 28 and the son will be 7, so the father will be 4 times as old as his son.]

ANS 4.

Clearly pattern is $1^2+0, 2^2+1, 3^2+2, \ldots$. So, number of tiles in tenth figure is $10^2+9=109$.

ANS 5.

$$=\left(\frac{1}{2-1}\times\frac{1}{2+1}\right)+\left(\frac{1}{4-1}\times\frac{1}{4+1}\right)+\ldots+\left(\frac{1}{20-1}\times\frac{1}{20+1}\right)$$

$$=\frac{1}{1.3}+\frac{1}{3.5}+\ldots$$

$$=\frac{1}{2}\left[\frac{(3-1)}{1.3}+\frac{(5-3)}{3.5}+\ldots\right]$$

$$=\frac{1}{2}\left[1-\frac{1}{3}+\frac{1}{3}-\frac{1}{5}+\ldots\frac{1}{19}-\frac{1}{21}\right]$$

$$=\frac{1}{2}\left[1-\frac{1}{21}\right]=\frac{10}{21}$$

ANS 6.

$$\sqrt{1+1\times2\times3\times4}=5=2\times3-1$$

$$\sqrt{1+2\times3\times4\times5}=11=3\times4-1$$

$$\sqrt{1+3\times4\times5\times6}=19=4\times5-1$$

$$\sqrt{1+4\times5\times6\times7}=29=5\times6-1$$

....

$$\sqrt{1+19\times20\times21\times22}=20\times21-1=419$$

ANS 7.

Solution 1
We use the fact that $g(x) = g(f(f^{-1}(x)))$.
Since $f(x) = 2x + 1$, then to determine $f^{-1}(x)$ we solve $x = 2y + 1$ for y to get $2y = x - 1$ or $y = \frac{1}{2}(x - 1)$. Thus, $f^{-1}(x) = \frac{1}{2}(x - 1)$.
Since $g(f(x)) = 4x^2 + 1$, then

$$\begin{aligned} g(x) &= g(f(f^{-1}(x))) \\ &= g(f(\tfrac{1}{2}(x-1))) \\ &= 4(\tfrac{1}{2}(x-1))^2 + 1 \\ &= 4 \cdot \tfrac{1}{4}(x-1)^2 + 1 \\ &= (x-1)^2 + 1 \\ &= x^2 - 2x + 2 \end{aligned}$$

Solution 2
We use the expressions for $f(x)$ and $g(f(x))$ to construct $g(x)$.
Since $f(x)$ is linear and $g(f(x))$ is quadratic, then it is likely that $g(x)$ is also quadratic.
Since $f(x) = 2x + 1$, then $(f(x))^2 = 4x^2 + 4x + 1$.
Since $g(f(x))$ has no term involving x, then we subtract $2f(x)$ (to remove the $4x$ term) to get

$$(f(x))^2 - 2f(x) = (4x^2 + 4x + 1) - 2(2x + 1) = 4x^2 - 1$$

To get $g(f(x))$ from this, we add 2 to get $4x^2 + 1$.
Therefore, $g(f(x)) = (f(x))^2 - 2f(x) + 2$, and so an expression for $g(x)$ is $x^2 - 2x + 2$.

ANS 8.

Solution 1
Since $ABCD$ is a rectangle, then $AB = CD = 40$ and $AD = BC = 30$.
By the Pythagorean Theorem, $BD^2 = AD^2 + AB^2$ and since $BD > 0$, then

$$BD = \sqrt{30^2 + 40^2} = \sqrt{900 + 1600} = \sqrt{2500} = 50$$

We calculate the area of $\triangle ADB$ is two different ways.
First, using AB as base and AD as height, we obtain an area of $\frac{1}{2}(40)(30) = 600$.
Next, using DB as base and AF as height, we obtain an area of $\frac{1}{2}(50)x = 25x$.
We must have $25x = 600$ and so $x = \frac{600}{25} = 24$.

Solution 2
Since $ABCD$ is a rectangle, then $AB = CD = 40$ and $AD = BC = 30$.
By the Pythagorean Theorem, $BD^2 = AD^2 + AB^2$ and since $BD > 0$, then

$$BD = \sqrt{30^2 + 40^2} = \sqrt{900 + 1600} = \sqrt{2500} = 50$$

Since $\triangle DAB$ is right-angled at A, then $\sin(\angle ADB) = \dfrac{AB}{BD} = \dfrac{40}{50} = \dfrac{4}{5}$.
But $\triangle ADF$ is right-angled at F and $\angle ADF = \angle ADB$.
Therefore, $\sin(\angle ADF) = \dfrac{AF}{AD} = \dfrac{x}{30}$.
Thus, $\dfrac{x}{30} = \dfrac{4}{5}$ and so $x = \frac{4}{5}(30) = 24$.

Solution 3
Since $ABCD$ is a rectangle, then $AB = CD = 40$ and $AD = BC = 30$.
By the Pythagorean Theorem, $BD^2 = AD^2 + AB^2$ and since $BD > 0$, then

$$BD = \sqrt{30^2 + 40^2} = \sqrt{900 + 1600} = \sqrt{2500} = 50$$

Note that $\triangle BFA$ is similar to $\triangle BAD$, since each is right-angled and they share a common angle at B.
Thus, $\dfrac{AF}{AB} = \dfrac{AD}{BD}$ and so $\dfrac{x}{30} = \dfrac{40}{50}$ which gives $x = \dfrac{30(40)}{50} = 24$.

ANS 9.

The number of integers between 100 and 999 inclusive is $999 - 100 + 1 = 900$.
An integer n in this range has three digits, say a, b and c, with the hundreds digit equal to a.
Note that $0 \le b \le 9$ and $0 \le c \le 9$ and $1 \le a \le 9$.
To have $a + b + c = 24$, then the possible triples for a, b, c in some order are 9,9,6; 9,8,7; 8,8,8. (There cannot be three 9's. If there are two 9's, the the other digit equals 6. If there is one 9, the second and third digits add to 15 but are both less than 9, so must equal 8 and 7. If there are zero 9's, the maximum for each digit is 8, and so each digt must be 8 in order for the sum of all three to equal 24.)
If the digits are 9, 9 and 6, there are 3 arrangements: 996, 969, 699.

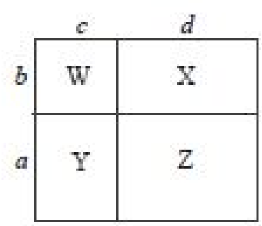
If the digits are 9, 8 and 7, there are 6 arrangements: 987, 978, 897, 879, 798, 789.
If the digits are 8, 8 and 8, there is only 1 arrangement: 888.
Therefore, there are $3 + 6 + 1 = 10$ integers n in the range 100 to 999 with the sum of the digits of n equal to 24.
The required probability equals the number of possible values of n with the sum of digits equal to 24 divided by the total number of integers in the range, or $\frac{10}{900} = \frac{1}{90}$.

ANS 10.

Clearly (D)

ANS 11.

Label the lengths of the vertical and horizontal segments as a, b, c, d, as shown.

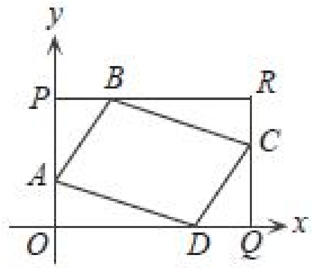


Rectangle W is b by c, so its perimeter is $2b + 2c$, which equals 2.
Rectangle X is b by d, so its perimeter is $2b + 2d$, which equals 3.
Rectangle Y is a by c, so its perimeter is $2a + 2c$, which equals 5.
Rectangle Z is a by d, so its perimeter is $2a + 2d$.
Therefore, $2a+2d = (2a+2b+2c+2d)-(2b+2c) = (2a+2c)+(2b+2d)-(2b+2c) = 5+3-2 = 6$.

ANSWER: (A)

ANS 12.

We draw a horizontal line through B (meeting the y-axis at P) and a vertical line through C (meeting the x-axis at Q). Suppose the point of intersection of these two lines is R.



We know that P has coordinates $(0,3)$ (since B has y-coordinate 3) and Q has coordinates $(5,0)$ (since C has x-coordinate 5), so R has coordinates $(5,3)$.
Using the given coordinates, $OA = 1$, $AP = 2$, $PB = 1$, $BR = 4$, $RC = 1$, $CQ = 2$, $QD = 1$,

and $DO = 4$.
The area of $ABCD$ equals the area of $PRQO$ minus the areas of triangles APB, BRC, CQD, and DOA.
$PRQO$ is a rectangle, so has area $3 \times 5 = 15$.
Triangles APB and CQD have bases PB and QD of length 1 and heights AP and CQ of length 2, so each has area $\frac{1}{2}(1)(2) = 1$.
Triangles BRC and DOA have bases BR and DO of length 4 and heights CR and AO of length 1, so each has area $\frac{1}{2}(4)(1) = 2$.
Thus, the area of $ABCD$ is $15 - 1 - 1 - 2 - 2 = 9$.
(Alternatively, we could notice that $ABCD$ is a parallelogram. Therefore, if we draw the diagonal AC, the area is split into two equal pieces. Dropping a perpendicular from C to Q on the x-axis produces a trapezoid $ACQO$ from which only two triangles need to be removed to determine half of the area of $ABCD$.)

ANS 13.

We make a chart to determine the sum of each possible combination of top faces. In the chart, the numbers across the top are the numbers from the first die and the numbers down the side are the numbers from the second die. For example, the number in the fourth column and fifth row is the sum of the fourth possible result from the first die and the fifth possible result from the second die, or $3 + 5 = 8$.

ANS 14.

$990 = 2 \times 3 \times 3 \times 5 \times 11$
The appearance of 11 in the above suggests that 11! will be the smallest n since 11 is prime.

	2	2	3	3	5	8
2	4	4	5	5	7	10
2	4	4	5	5	7	10
3	5	5	6	6	8	11
3	5	5	6	6	8	11
5	7	7	8	8	10	13
8	10	10	11	11	13	16

So the possibilities are 4, 5, 6, 7, 8, 10, 11, 13, 16, or nine possibilities in total.

ANS 15. Clearly (A)

SECTION – B

ANS 16.

Since $S = \sqrt{x_1 + x_2 - x_3 - x_4}$ is a real number, then $x_1 + x_2 - x_3 - x_4 \geq 0$ which means $x_1 + x_2 \geq x_3 + x_4$.

Also, $x_1 + x_2 + x_3 + x_4 \geq 10$.

Therefore, $x_1 + x_2 \geq 5$.

If $x_1 + x_2 = 5$, then there are 4 ways to do so. $(x_1, x_2) = (1,4)$ or $(4,1)$ or $(2,3)$ or $(3,2)$

If $x_1 + x_2 = 6$, then there are 2 ways. $(x_1, x_2) = (2,4)$ or $(4,2)$

If $x_1 + x_2 = 7$, then there are 2 ways. $(x_1, x_2) = (3,4)$ or $(4,3)$

This gives a total of 8 ways.

Each case gives rise to 2 possibilities for x_3 and x_4 (either order).

Altogether, there are 16 possibilities.

ANS 17.

Solution 1

First, we add the two given equations to obtain

$$(f(x)+g(x))+(f(x)-g(x))=(3x+5)+(5x+7)$$

or $2f(x)=8x+12$ which gives $f(x)=4x+6$.
Since $f(x)+g(x)=3x+5$, then $g(x)=3x+5-f(x)=3x+5-(4x+6)=-x-1$.
(We could also find $g(x)$ by subtracting the two given equations or by using the second of the given equations.)
Since $f(x)=4x+6$, then $f(2)=14$.
Since $g(x)=-x-1$, then $g(2)=-3$.
Therefore, $2f(2)g(2)=2\times 14\times(-3)=-84$.

Solution 2

Since the two given equations are true for all values of x, then we can substitute $x=2$ to obtain

$$\begin{aligned} f(2)+g(2) &= 11 \\ f(2)-g(2) &= 17 \end{aligned}$$

Next, we add these two equations to obtain $2f(2)=28$ or $f(2)=14$.
Since $f(2)+g(2)=11$, then $g(2)=11-f(2)=11-14=-3$.
(We could also find $g(2)$ by subtracting the two equations above or by using the second of these equations.)
Therefore, $2f(2)g(2)=2\times 14\times(-3)=-84$.

ANS 18.

Let $\dfrac{x^2+14x+9}{x^2+2x+3}=y$

Cross multiplying and rearranging the terms, we get

$x^2(1-y)+x(14-2y)+(9-3y)=0$ ..(1)

here, discriminant, $D=(14-2y)^2-4(1-y)(9-3y)$

$=160-8y^2-8y$

$=-8(y^2+y-20)$

$=-8(y+5)(y-4)$

Now, since x is real, hence two roots of equation (1) are real. Hence, $D\geq 0$.

$\Rightarrow\quad -8(y+5)(y-4)\geq 0$

$\Rightarrow\quad (y+5)(y-4)\leq 0$

$\Rightarrow\quad$ either $y+5\geq 0$ and $y-4\leq 0$ ...(2)

or $\quad y+5\leq 0$ and $y-4\geq 0$ ...(3)

from eqn. (2), $y\geq -5$ and $y\leq 4 \Rightarrow -5\leq y\leq 4$

from eqn. (3), $y\leq -5$ and $y\geq 4$, which is not possible.

$\therefore\quad -5\leq y\leq 4$.

$\therefore\quad$ least value = – **5** and greatest value = **4.**

ANS 19.

Suppose the second car overtakes the first car after t hours. Then, the two cars travel the same distance in t hours.

Distance travelled by the first car in t hours = $10\,t$ km.

Distance travelled by the second car in t hours

$$= \text{sum of } t \text{ terms of an A.P. with } a = 8,\ d = \frac{1}{2}.$$

$$= \frac{t}{2}\left[2 \times 8 + (t-1)\frac{1}{2}\right] = \frac{t}{4}(t+31)$$

according to question, $10t = \frac{t}{4}(t+31)$

$\Rightarrow \quad t^2 - 9\,t = 0 = t = 0 \quad \text{or} \quad t = 9 \Rightarrow t = 9$ **hours** $(\because \quad t \neq 0)$

ANS 20.

Solution:

The best way to visualize the problem is to unfold the box and draw its net in two dimensions (see picture). The straight line connecting points A and B is the shortest path connecting opposite vertices. Using the Pythagorean theorem $\sqrt{4^2 + 5^2} = \sqrt{41}$, is the length of the shortest path. Notice that one may need to unfold the net three different ways to find the actual shortest path represented by a straight line segment. The correct answer is $\sqrt{41}$