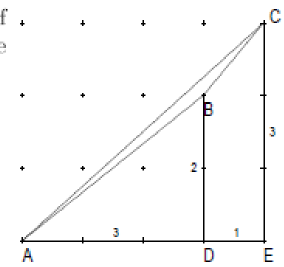
THIRD InTRa ScHool MaTHeMaTIcS olyMpIaD 2012

CLASS X

SolutionS

1

There are various constructions that can be used to find the area of ΔABC. One possibility is shown in the figure, which also shows the lengths of the various sides.
The area of triangle ACE is $\frac{1}{2}(4)(3) = 6$.
The area of triangle ABD is $\frac{1}{2}(3)(2) = 3$.
The area of trapezium $BDEC$ is $\frac{1}{2}(2+3)(1) = 5/2$.
Therefore the area of ΔABC is $6 - 3 - 5/2 = 1/2$.



Answer: B

2

10^{20} when written out in full consists of a one followed by 20 zeroes. Therefore $10^{20} - 2$ consists of 19 nines and ends in an eight. The sum of these digits is $19 \times 9 + 8 = 20 \times 9 - 1 = 179$.

Answer: E

3

The lowest common multiple of 8 and 5 is 40. Therefore they will be off together every 40th day. The year 2000 has 366 days. Hence 31 December 2000 is 366-20 = 346 days after 20 January. We need the highest multiple of 40 which is less than 346. That is 320 which is 26 less than 346. The required date then is 31-26=5. The 5th day of December.

Answer: B

4

$2001 \times \frac{1}{2001} = 1 < 2001 - \frac{1}{2001} < 2001 + \frac{1}{2001} < 2001, 2001 < 2001 \div \frac{1}{2001} = 2001 \times 2001$.

Answer: B

5

Answer B. We can draw up a table of the first few remainders and hope to find a pattern.

n	1	2	3	4	5	6	7	...
2^n	2	4	8	16	32	64	128	...
Rem.	2	4	1	2	4	1	2	...

It certainly appears that there are only three distinct remainders (1, 2, and 4), which repeat cyclically, and in fact modular arithmetic (the arithmetic of remainders) guarantees that the pattern does continue unchanged. However, if you would like to prove that (say) 4 is always followed by 1, suppose 2^n has remainder 4 after dividing by 7. Then $2^n = 7q + 4$, where q is the quotient. It follows that

$$2^{n+1} = 2 \times 2^n = 2 \times (7q + 4) = 14q + 8 = 7(2q + 1) + 1,$$

so the next remainder is indeed 1, as we expected.

6

Answer C. Triangle ABC is isosceles, so angles $\widehat{A}$ and $\widehat{B}$ are equal. Also $Q\widehat{P}B = \widehat{A}$ and $M\widehat{P}A = \widehat{B}$ (corresponding angles), so triangles APM and PBQ are also isosceles. Thus $MP = MA$ and $PQ = BQ$ so

$$CM + MP + PQ + QC = CM + MA + BQ + QC = CA + BC = 15 + 15 = 30.$$

Alternatively, since the position of P is unspecified, you can make it coincide with B (say). The perimeter is then $2BC = 30$.

7(A)

8 (B)

9(A)

10 (A)

11

. *Solution 1*

We know $p + \cfrac{1}{q + \cfrac{1}{r}} = \frac{25}{19} = 1 + \frac{6}{19} = 1 + \frac{1}{\frac{19}{6}} = 1 + \cfrac{1}{3 + \cfrac{1}{6}}$.

Therefore, comparing the two fractions, $p = 1$, $q = 3$ and $r = 6$.

Solution 2

Since p, q and r are positive integers, then $q + \frac{1}{r}$ is at least 1, so $\cfrac{1}{q + \cfrac{1}{r}}$ is between 0 and 1.

Since $p + \cfrac{1}{q + \cfrac{1}{r}}$ is equal to $\frac{25}{19}$ which is between 1 and 2, then p must be equal to 1.

Therefore, $\cfrac{1}{q + \cfrac{1}{r}} = \frac{25}{19} - 1 = \frac{6}{19}$ or $q + \frac{1}{r} = \frac{19}{6}$.

Since r is a positive integer, then $\frac{1}{r}$ is between 0 and 1, so since $\frac{19}{6}$ is between 3 and 4, then $q = 3$.

(We are not asked to determine what the value of r, but we can check that $r = 6$.)

ANSWER: (C)

12

The differences between successive terms are 2, 3, 4, 5, 6, 7, 8, 9, 10, and so on. Writing out some more triangular numbers we will notice a pattern,

1	3	6	10	15
21	28	36	45	55
66	78	91	105	120
136	153			and so on.

The integers in the last two vertical columns are always divisible by 5. So in each horizontal row 2 out of the 5 integers are divisible by 5. Two fifths of 250 is 100.

Answer: A

[Why are the integers in the last two columns always divisible by 5? Notice that each triangular number can be written as the sum of numbers:

$$\begin{aligned} 1 &= 1 \\ 3 &= 1+2 \\ 6 &= 1+2+3 \\ 10 &= 1+2+3+4 \\ 15 &= 1+2+3+4+5, \end{aligned}$$

and so on.

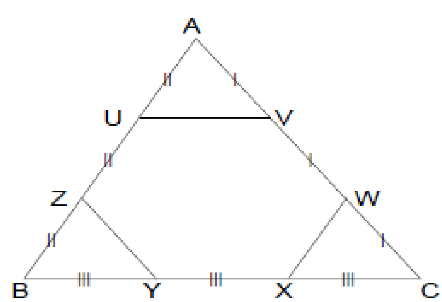
From this pattern, or by any of several of other methods, you can observe that

$$\begin{aligned} 1 &= \tfrac{1}{2}(1\times 2), \\ 3 &= \tfrac{1}{2}(2\times 3), \\ 6 &= \tfrac{1}{2}(3\times 4), \\ 10 &= \tfrac{1}{2}(4\times 5), \\ 15 &= \tfrac{1}{2}(5\times 6), \end{aligned}$$

and so on. The n-th triangular number is $\frac{1}{2}n(n+1)$. Clearly $\frac{1}{2}n(n+1)$ is divisible by 5 whenever n or its successor $n+1$, is divisible by 5.]

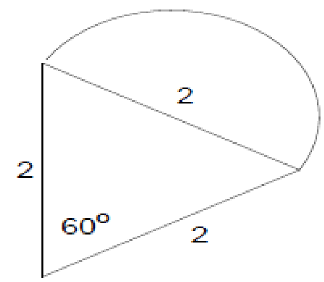
13

The area of $\triangle AUV$ is $\frac{1}{2}(AU)\times$ (the height of $\triangle AUV$). But AU is one third of AB, and the height of $\triangle AUV$ is one third the height of $\triangle ABC$. Therefore the area of $\triangle AUV$ is $(\frac{1}{3})^2 = \frac{1}{9}$ of the total area of $\triangle ABC$. Similarly the areas of $\triangle BZY$ and $\triangle CXW$ are $\frac{1}{9}$ of the area of $\triangle ABC$. By removing the three triangles we obtain the area of the hexagon $UVWXYZ$ to be $(1-\frac{3}{9}) = \frac{2}{3}$ of $\triangle ABC$.



14

Area of a single 'ice cream cone' : $\frac{1}{2}\pi + \frac{1}{2} \times 2 \times 2\sin(60^o) = \frac{1}{2}(\pi + 2\sqrt{3})$. The total area consists of six 'cones' and is given by $\frac{1}{3}(\pi + 2\sqrt{3})$.



15

The first newcomer needs 20 lines and the second one needs 21.

<u>Answer</u>: D

16

Answer E. The simplest method is to try out each of the given remainders to see which of them gives the right remainders after division by $x - 1$ and $x - 2$. More rigorously, after division by $(x - 1)(x - 2)$ the remainder must be of the form $ax + b$, so if the quotient is $q(x)$, then the polynomial is $(x - 1)(x - 2)q(x) + ax + b$. This can be written as $(x - 1)\{(x - 2)q(x) + a\} + a + b$, or as $(x - 2)\{(x - 1)q(x) + a\} + 2a + b$, so $a + b = 2$ and $2a + b = 1$, giving $a = -1$ and $b = 3$. (The remainders can also be found by the Remainder Theorem.)

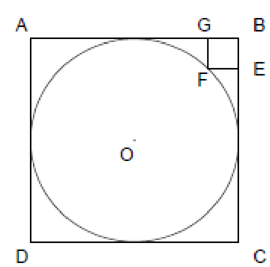
<u>Answer</u>: E

17

Answer A. Suppose the first bricklayer lays x bricks per hour on his own, and the second one lays y bricks per hour. Then the number of bricks in the wall is $9x$ which is equal to $10y$. Working together, they lay $x + y - 10$ bricks per hour, so the number of bricks in the wall is also $5(x + y - 10)$. Solving the equations gives $x = 100$ and $y = 90$, so the number of bricks in the wall is 900.

18

Let R be the radius of the circle. Then AB has length 2R, and OB is $\sqrt{2}R$. But OB is also $OF + FB = R + \sqrt{2}$, because FB is the diameter of a square of side 1. Therefore $\sqrt{2}R = R + \sqrt{2}$, and solving for R we obtain $R(\sqrt{2} - 1) = \sqrt{2}$, so that $R = \sqrt{2}/(\sqrt{2} - 1)$, and
$AB = 2R = \frac{2\sqrt{2}}{\sqrt{2}-1} \times \frac{\sqrt{2}+1}{\sqrt{2}+1} = 4 + 2\sqrt{2}$.



Answer: A

19)

Answer E. Let the three points of tangency be P (on AB produced), Q (on AC produced), and R (on BC). Let $P\widehat{O}B = \alpha$ and $Q\widehat{O}C = \beta$. Then $R\widehat{O}B = \alpha$, because the right-angled triangles POB and ROB are congruent, and similarly $R\widehat{O}C = \beta$. We can now obtain the result in at least two different ways.

(a) It follows that $A\widehat{B}C = 2\alpha$ and $A\widehat{C}B = 2\beta$, so $2(\alpha + \beta) = 180° - \widehat{A} = 150°$. Finally, $B\widehat{O}C = \alpha + \beta = 75°$.

(b) Note that $2B\widehat{O}C = 2\alpha + 2\beta = P\widehat{O}Q$ regardless of the positions of B and C. This means that $B\widehat{O}C = A\widehat{O}Q$ (just move B to coincide with A in which case C coincides with Q). Now it is straightforward to find that $A\widehat{O}Q = 90° - 15° = 75°$.

20)