ANSWER KEY

Mathematics class XII First Term 2012-13

Section-A

① $a * b = 2a + b$

$\Rightarrow (2*3)*4 = (2\times2+3)*4 = 7*4 = 2\times7+4 = \underline{18}$ Ans

② $\sin\left(\frac{\pi}{3} + \sin^{-1}\frac{1}{2}\right) = \sin\left(\frac{\pi}{3}+\frac{\pi}{6}\right) = \sin\frac{\pi}{2} = \underline{1}$ Ans.

③ Put $x = \sec\theta \Rightarrow \theta = \sec^{-1}x$

$\therefore \cot^{-1}\left(\frac{1}{\sqrt{x^2-1}}\right) = \cot^{-1}\left(\frac{1}{\sqrt{\sec^2\theta-1}}\right) = \cot^{-1}\left(\frac{1}{\tan\theta}\right) = \cot^{-1}(\cot\theta) = \theta$

$= \underline{\sec^{-1}x}$ Ans

④ $\text{adj } AB = (\text{adj } B).(\text{adj } A)$

$= \begin{pmatrix} 1 & -2 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 4 & -1 \end{pmatrix} = \begin{pmatrix} 2-8 & 3+2 \\ -6+4 & -9-1 \end{pmatrix} = \underline{\begin{pmatrix} -6 & 5 \\ -2 & -10 \end{pmatrix}}$ Ans.

⑤ $(a+ib)(a-ib) - (c+id)(-c+id)$

$= (a^2 - i^2b^2) - (i^2d^2 - c^2) = \underline{a^2+b^2+c^2+d^2}$ Ans

⑥ $y = \frac{\log x}{\log 7} \quad \therefore \frac{dy}{dx} = \frac{1}{x.\log 7}$ Ans.

⑦ $y = e^{\sin x} \Rightarrow \frac{dy}{dx} = e^{\sin x}\times\cos x \quad \therefore \left.\frac{dy}{dx}\right]_{x=\pi/2} = e^{\sin\pi/2}\times\cos\frac{\pi}{2} = \underline{0}$ Ans.

⑧ $\frac{dy}{dx} = 3x^2-3 \quad \therefore$ slope of normal $= \frac{-1}{3x^2-3} = \frac{-1}{3(2)^2-3} = \underline{\frac{-1}{9}}$ Ans.

⑨ $\int \frac{\cos^2x - \sin^2x}{\sin^2x\,\cos^2x}\,dx = \int\left(\frac{1}{\sin^2x} - \frac{1}{\cos^2x}\right)dx = \int(\operatorname{cosec}^2x - \sec^2x)\,dx$

$= \underline{-\cot x - \tan x + c}$ Ans.

⑩ Put $1+x^3 = t \Rightarrow 3x^2dx = dt \Rightarrow x^2dx = \frac{dt}{3}$

$\therefore I = \frac{1}{3}\int\frac{dt}{t} = \frac{1}{3}\log|t| + c$

$= \frac{1}{3}\log|1+x^3| + c$ Ans.

Section - B

Ans 16. $A' = \begin{pmatrix} 3 & 4 & 0 \\ 2 & 1 & 6 \\ 5 & 3 & 7 \end{pmatrix}$

Let $B = \frac{1}{2}(A + A') = \frac{1}{2}\left[\begin{pmatrix} 3 & 2 & 5 \\ 4 & 1 & 3 \\ 0 & 6 & 7 \end{pmatrix} + \begin{pmatrix} 3 & 4 & 0 \\ 2 & 1 & 6 \\ 5 & 3 & 7 \end{pmatrix}\right] = \frac{1}{2}\begin{pmatrix} 6 & 6 & 5 \\ 6 & 2 & 9 \\ 5 & 9 & 14 \end{pmatrix}$

here, $B' = \frac{1}{2}\begin{pmatrix} 6 & 6 & 5 \\ 6 & 2 & 9 \\ 5 & 9 & 14 \end{pmatrix} = B \Rightarrow B$ is symmetric matrix.

Let $C = \frac{1}{2}(A - A') = \frac{1}{2}\left[\begin{pmatrix} 3 & 2 & 5 \\ 4 & 1 & 3 \\ 0 & 6 & 7 \end{pmatrix} - \begin{pmatrix} 3 & 4 & 0 \\ 2 & 1 & 6 \\ 5 & 3 & 7 \end{pmatrix}\right] = \frac{1}{2}\begin{pmatrix} 0 & -2 & 5 \\ 2 & 0 & -3 \\ -5 & 3 & 0 \end{pmatrix}$

here, $C' = \frac{1}{2}\begin{pmatrix} 0 & 2 & -5 \\ -2 & 0 & 3 \\ 5 & -3 & 0 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 0 & -2 & 5 \\ 2 & 0 & -3 \\ -5 & 3 & 0 \end{pmatrix} = -C \Rightarrow C$ is skew-symm.

Now, $A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A') = B + C$

$$= \frac{1}{2}\begin{pmatrix} 6 & 6 & 5 \\ 6 & 2 & 9 \\ 5 & 9 & 14 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & -2 & 5 \\ 2 & 0 & -3 \\ -5 & 3 & 0 \end{pmatrix}$$

Ans 17. clearly, when $a, b \in N$ then $\frac{a+b}{2} \notin N$ always.

eg. $a = 1$, $b = 2$ then $\frac{1+2}{2} = \frac{3}{2} \notin N$ $\therefore$ $*$ is not binary operation in this case.

when, $a, b \in Q$ then clearly $*$ is binary operation.

for commutative

$$b * a = \frac{b+a}{2} = \frac{a+b}{2} = a * b \quad \forall\ a, b \in Q$$

$\Rightarrow *$ is commutative.

for associative

let $a, b, c \in Q$

Consider, $(a * b) * c = \left(\frac{a+b}{2}\right) * c = \frac{1}{2}\left(\frac{a+b}{2} + c\right) = \frac{1}{4}(a+b+2c)$

also, $a * (b * c) = a * \left(\frac{b+c}{2}\right) = \frac{1}{2}\left(a + \frac{b+c}{2}\right) = \frac{1}{4}(2a+b+c)$

clearly, $(a * b) * c \neq a * (b * c) \Rightarrow *$ is not associative.

OR

$R = \{(a,b) : 4 \text{ divides } a-b;\ a,b \in Z\}$

For reflexive: clearly 4 divides $a-a$ i.e. 0.

$\Rightarrow (a,a) \in R$

For symmetric: let $(a,b) \in R$

$\Rightarrow$ 4 divides $a-b$

$\Rightarrow$ 4 also divides $-(a-b)$.

$\Rightarrow$ 4 divides $b-a$

$\Rightarrow (b,a) \in R$. $\therefore$ R is symmetric also.

For Transitive: let $a,b,c \in Z$.

Let $(a,b) \in R$ and $(b,c) \in R$

Since, $(a,b) \in R \Rightarrow$ 4 divides $a-b$. — ①

Since $(b,c) \in R \Rightarrow$ 4 divides $b-c$. — ②

from ① and ②, 4 divides $(a-\cancel{b}) + (\cancel{b}-c)$ also.

$\Rightarrow$ 4 divides $(a-c)$ also.

$\Rightarrow (a,c) \in R \Rightarrow$ R is transitive also.

Since R is reflexive, transitive & symmetric, $\therefore$ R is an equivalence relation. Hence shown.

Ans 13. $\tan^{-1}(x-1) + \tan^{-1}(x+1) = \tan^{-1}3x - \tan^{-1}x$

$$\Rightarrow \tan^{-1}\left[\frac{(x-1)+(x+1)}{1-(x-1)(x+1)}\right] = \tan^{-1}\left[\frac{3x-x}{1+(3x)(x)}\right]$$

when $x^2-1 < 1$ and $3x^2 > -1$

$$\Rightarrow \tan^{-1}\left(\frac{2x}{1-x^2+1}\right) = \tan^{-1}\left(\frac{2x}{1+3x^2}\right)$$

$$\Rightarrow \frac{\cancel{2}x}{2-x^2} = \frac{\cancel{2}x}{1+3x^2} \Rightarrow x(1+3x^2) - x(2-x^2) = 0$$

$\Rightarrow x[1+3x^2-2+x^2] = 0 \Rightarrow x(4x^2-1) = 0$

$\Rightarrow x = 0$ or $x^2 = \frac{1}{4} \Rightarrow x = \pm\frac{1}{2}$

$\therefore x = 0, \pm\frac{1}{2}$ as all values satisfies given equation.

OR

$$LHS = \sin^{-1}\left(\frac{8}{17}\times\sqrt{1-\left(\frac{3}{5}\right)^2} + \frac{3}{5}\times\sqrt{1-\left(\frac{8}{17}\right)^2}\right) = \sin^{-1}\left(\frac{8}{17}\times\frac{4}{5} + \frac{3}{5}\times\frac{15}{17}\right) = \sin^{-1}\frac{77}{85}$$

$$= \cos^{-1}\frac{36}{85} = RHS.$$

Ans 14. LHS. using $C_2 \to C_1 - C_2$, $C_3 \to C_1 - C_3$

$$\begin{vmatrix} 1 & 0 & 0 \\ a & a-b & a-c \\ a^3 & a^3-b^3 & a^3-c^3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ a & a-b & a-c \\ a^3 & (a-b)(a^2+b^2+ab) & (a-c)(a^2+c^2+ac) \end{vmatrix}$$

$$= (a-b)(a-c)\begin{vmatrix} 1 & 0 & 0 \\ a & 1 & 1 \\ a^3 & a^2+b^2+ab & a^2+c^2+ac \end{vmatrix}$$

$C_2 \to C_2 - C_3$

$$= (a-b)(a-c)\begin{vmatrix} 1 & 0 & 0 \\ a & 0 & 1 \\ a^3 & b^2+ab-c^2-ac & a^2+c^2+ac \end{vmatrix}$$

$$= (a-b)(a-c)\begin{vmatrix} 1 & 0 & 0 \\ a & 0 & 1 \\ a^3 & (b-c)(a+b+c) & a^2+c^2+ac \end{vmatrix}$$

$$= (a-b)(a-c)\{-(b-c)(a+b+c)\} = (a-b)(b-c)(c-a)(a+b+c) = \text{RHS}.$$

Ans 15. Since f is cts at $x = 1$.

$$\therefore \lim_{h \to 0} f(1-h) = \lim_{h \to 0} f(1+h) = f(1)$$

$$\Rightarrow \lim_{h \to 0} [5a(1-h) - 2b] = \lim_{h \to 0} [3a(1+h) + b] = 11$$

$\Rightarrow 5a - 2b = 11$ and $3a + b = 11$. solving, $\underline{a = 3}$, $\underline{b = 2}$ Ans

Ans 16. $x = \dfrac{\cos y}{\cos(a+y)}$, diff both sides w.r.t. y,

$$\frac{dx}{dy} = \frac{\cos(a+y)\cdot(-\sin y) - \cos y\cdot(-\sin(a+y))}{\cos^2(a+y)}$$

$$= \frac{\sin(a+y)\cos y - \sin y\cos(a+y)}{\cos^2(a+y)} = \frac{\sin(a+\cancel{y}-\cancel{y})}{\cos^2(a+y)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a} \quad \text{H. proved.}$$

Ans 17. $\dfrac{dy}{dx} = 3(-\sin(\log x)) \times \dfrac{1}{x} + 4\cos(\log x) \times \dfrac{1}{x}$

$$\Rightarrow x\frac{dy}{dx} = -3\sin(\log x) + 4\cos(\log x)$$

diff again,

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx}.1 = -3\cos(\log x)\times\frac{1}{x} + 4(-\sin(\log x))\times\frac{1}{x}$$

$$\Rightarrow x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = -(3\cos(\log x) + 4\sin(\log x))$$

$$= -y$$

$$\Rightarrow x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$$. H. proved.

OR

taking log both sides,

$$y.\log(\cos x) = x.\log(\sin y)$$

diff both sides w.r.t. x,

$$y.\frac{1}{\cos x}(-\sin x) + \log(\cos x).\frac{dy}{dx} = x.\frac{1}{\sin y}.\cos y.\frac{dy}{dx} + \log(\sin y).1$$

$$\Rightarrow -y\tan x - \log(\sin y) = [x\cot y - \log(\cos x)]\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{\log(\cos x) - x\cot y}{y\tan x + \log(\sin y)}\right)^{-1} = \frac{y\tan x + \log(\sin y)}{\log(\cos x) - x\cot y}$$ Ans.

Ans 18. $$\frac{dy}{d\theta} = \frac{(2+\cos\theta)(4\cos\theta) - 4\sin\theta(-\sin\theta)}{(2+\cos\theta)^2} - 1$$

$$= \frac{8\cos\theta + 4\cos^2\theta + 4\sin^2\theta}{(2+\cos\theta)^2} - 1$$

$$= \frac{8\cos\theta + 4 - (2+\cos\theta)^2}{(2+\cos\theta)^2} = \frac{8\cos\theta + 4 - 4 - \cos^2\theta - 4\cos\theta}{(2+\cos\theta)^2}$$

$$= \frac{4\cos\theta - \cos^2\theta}{(2+\cos\theta)^2} = \frac{\cos\theta(4-\cos\theta)}{(2+\cos\theta)^2}$$

Since $\theta \in [0, \frac{\pi}{2}] \Rightarrow \cos\theta > 0$.

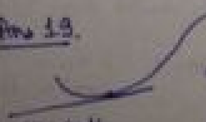
also, $0 \leq \cos\theta \leq 1 \Rightarrow 0 \geq -\cos\theta \geq -1 \Rightarrow 4 \geq 4 - \cos\theta \geq 3$

$\Rightarrow 3 \leq 4 - \cos\theta \leq 4 \Rightarrow 4 - \cos\theta > 0$

also, $(2+\cos\theta)^2 > 0$ always. $\therefore \frac{dy}{d\theta} = \frac{(+)\times(+)}{(+)} \Rightarrow \frac{dy}{d\theta} > 0$

$\Rightarrow y$ is an increasing fn. Hence proved.

Ans 19.



$y = x - 11$

$y = x^3 - 11x + 5$

slope of tangent $= \frac{dy}{dx} = 3x^2 - 11$

also, slope of tangent $= 1$

acc. to que, $3x^2 - 11 = 1 \Rightarrow 3x^2 = 12 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$

Case I $x = 2$ then $y = (2)^3 - 11(2) + 5 = 8 - 22 + 5 = -9$

$\therefore$ point is $(2, -9)$.

Case II $x = -2$ then $y = (-2)^3 - 11(-2) + 5 = -8 + 22 + 5 = 19$

$\therefore$ point is $(-2, 19)$.

But $(-2, 19)$ do not satisfy $y = x - 11$. $\therefore$ we reject it.

$\therefore$ Required point is $\underline{(2, -9)}$ Ans.

Ans 20. $I = \int \frac{x+2}{\sqrt{x^2 - 5x + 6}}\, dx$

Put $x + 2 = A(2x - 5) + B$

$= 2Ax - 5A + B$

Comparing, $2A = 1$ & $-5A + B = 2$

$\Rightarrow A = \frac{1}{2}$, $\Rightarrow B = 2 + \frac{5}{2} = \frac{9}{2}$

$\therefore I = \int \frac{\frac{1}{2}(2x-5) + \frac{9}{2}}{\sqrt{x^2 - 5x + 6}}\, dx = \frac{1}{2}\int \frac{2x-5}{\sqrt{x^2-5x+6}}\, dx + \frac{9}{2}\int \frac{dx}{\sqrt{x^2-5x+6}}$

$= \frac{1}{2} I_1 + \frac{9}{2} I_2$ ——— (1)

for I_1, Put $x^2 - 5x + 6 = t \Rightarrow (2x - 5)\,dx = dt$

$\therefore I_1 = \int \frac{dt}{\sqrt{t}} = 2\sqrt{t} + C_1 = 2\sqrt{x^2 - 5x + 6} + C_1$ ——— (2)

for I_2, $x^2 - 5x + 6 = x^2 - 5x + \frac{25}{4} - \frac{25}{4} + 6 = \left(x - \frac{5}{2}\right)^2 - \left(\frac{1}{2}\right)^2$

$\therefore I_2 = \int \frac{dx}{\sqrt{\left(x - \frac{5}{2}\right)^2 - \left(\frac{1}{2}\right)^2}} = \log\left|\left(x - \frac{5}{2}\right) + \sqrt{x^2 - 5x + 6}\right| + C_2$ ——— (3)

from (1), (2) & (3),

$I = \frac{1}{2} \times 2\sqrt{x^2 - 5x + 6} + C_1 + \frac{9}{2}\log\left|\left(x - \frac{5}{2}\right) + \sqrt{x^2 - 5x + 6}\right| + C_2$

$= \sqrt{x^2 - 5x + 6} + \frac{9}{2}\log\left|\left(x - \frac{5}{2}\right) + \sqrt{x^2 - 5x + 6}\right| + C$ where $C_1 + C_2 = C$

Ans 21. $I = \frac{1}{2}\int (2\sin x \sin 3x)\cdot \sin 2x\, dx = \frac{1}{2}\int (\cos 2x - \cos 4x)\cdot \sin 2x\, dx$

$= \frac{1}{2}\int (\sin 2x \cos 2x - \sin 2x \cos 4x)\, dx = \frac{1}{4}\int (2\sin 2x \cos 2x - 2\sin 2x \cdot \cos 4x)\, dx$

$= \frac{1}{4}\int [\sin 4x - \sin 6x - \sin(-2x)]\, dx$

$= \frac{1}{4}\int(\sin 2x + \sin 4x - \sin 6x)\,dx$

$= \frac{1}{4}\left[-\frac{\cos 2x}{2} - \frac{\cos 4x}{4} + \frac{\cos 6x}{6}\right] + c$

$= -\frac{1}{48}(6\cos 2x + 3\cos 4x - 2\cos 6x) + c$ Ans.

Ans 22. $I = \int \frac{x.\sin^{-1}x}{\sqrt{1-x^2}}\,dx$

Put $x = \sin\theta \Rightarrow dx = \cos\theta\, d\theta$

$\therefore I = \int \frac{\sin\theta . \sin^{-1}(\sin\theta)}{\sqrt{1-\sin^2\theta}} \times \cos\theta\, d\theta$

$= \int \underset{I}{\theta} . \underset{II}{\sin\theta}\, d\theta$

$= \theta.(-\cos\theta) - \int 1.(-\cos\theta)\, d\theta + c$

$= -\theta\cos\theta + \sin\theta + c$

$= -\sin^{-1}x \times \sqrt{1-x^2} + x + c = \underline{x - \sqrt{1-x^2}.\sin^{-1}x + c}$ Ans.

OR

$I = \int\left(\frac{2\sin 2x\cos 2x - 4}{2\sin^2 2x}\right).e^x dx = \int\left(\frac{\sin 2x\cos 2x - 2}{\sin^2 2x}\right).e^x\,dx$

$= \int(\cot 2x - 2\operatorname{cosec}^2 2x)\,e^x\,dx$

$= \cot 2x\,.e^x + c$ $\quad (\because \int[f(x) + f'(x)].e^x dx = f(x).e^x + c$ here $f(x) = \cot 2x)$

Section-C

Ans 23. Given system of equations can be written as $AX = B$ where,

$A = \begin{pmatrix} 1 & 2 & -3 \\ 2 & 3 & 2 \\ 3 & -3 & -4 \end{pmatrix}$, $B = \begin{pmatrix} -4 \\ 2 \\ 11 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

Now, $|A| = 1(-12+6) - 2(-14) - 3(-15) = 67 \neq 0 \Rightarrow A^{-1}$ exist.

Now, $C_{11} = -6$, $C_{12} = 14$, $C_{13} = -15$

$C_{21} = 17$, $C_{22} = 5$, $C_{23} = 9$

$C_{31} = 13$, $C_{32} = -8$, $C_{33} = -1$

$\therefore \text{adj } A = \begin{pmatrix} -6 & 17 & 13 \\ 14 & 5 & -8 \\ -15 & 9 & -1 \end{pmatrix}$

Now, $A^{-1} = \frac{1}{|A|} \cdot \text{adj } A = \frac{1}{67}\begin{pmatrix} -6 & 17 & 13 \\ 14 & 5 & -8 \\ -15 & 9 & -1 \end{pmatrix}$

Now, $AX = B$

$\Rightarrow X = A^{-1}B$

$= \frac{1}{67}\begin{pmatrix} -6 & 17 & 13 \\ 14 & 5 & -8 \\ -15 & 9 & -1 \end{pmatrix}\begin{pmatrix} -4 \\ 2 \\ 11 \end{pmatrix} = \frac{1}{67}\begin{pmatrix} 24+34+143 \\ -56+10-88 \\ 60+18-11 \end{pmatrix} = \frac{1}{67}\begin{pmatrix} 201 \\ -134 \\ 67 \end{pmatrix}$

$= \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} \quad \therefore \quad \underline{x = 3, \ y = -2, \ z = 1} \text{ Ans.}$

<u>Ans 24</u>.

$V \to$ max. given $\to l$ when $\alpha = \tan^{-1}\sqrt{2}$

$V = \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi(l^2 - h^2) \cdot h = \frac{1}{3}\pi(l^2 h - h^3)$

$\frac{dV}{dh} = \frac{1}{3}\pi(l^2 - 3h^2)$

for max/min, $\frac{dV}{dh} = 0 \Rightarrow l^2 = 3h^2 \Rightarrow l = \sqrt{3}h$

also, $\frac{d^2V}{dh^2} = -2\pi h < 0 \Rightarrow V$ is max. when $l = \sqrt{3}h$ i.e. $h = \frac{l}{\sqrt{3}}$

$= -2\pi \frac{l}{\sqrt{3}} < 0$

Now, $r^2 = l^2 - \frac{l^2}{3} = \frac{2l^2}{3} \Rightarrow r = \frac{\sqrt{2}}{\sqrt{3}}l$

Also, $\tan\alpha = \frac{r}{h} = \frac{\sqrt{2}}{\sqrt{3}}l \div \frac{l}{\sqrt{3}} = \sqrt{2} \Rightarrow \alpha = \tan^{-1}\sqrt{2}$. H. Proved

<u>OR</u>

$R^2 = x^2 + r^2$. Now, Volume of cone $= V = \frac{1}{3}\pi r^2 h$

$\Rightarrow V = \frac{1}{3}\pi(R^2 - x^2)(R + x)$

$= \frac{1}{3}\pi(R+x)^2 \cdot (R - x)$

$\frac{dV}{dx} = \frac{1}{3}\pi\left[(R+x)^2(-1) + (R-x) \cdot 2(R+x)\right]$

$= \frac{1}{3}\pi(R+x)\left[-(R+x) + 2(R-x)\right] = \frac{1}{3}\pi(R+x)(R-3x)$

for max/min $\frac{dV}{dx} = 0 \Rightarrow \underset{\text{not possible}}{R + x = 0}$ or $R - 3x = 0$

$\Rightarrow R = 3x \Rightarrow x = \frac{R}{3}$

Now, $\frac{d^2V}{dx^2} = \frac{1}{3}\pi\left[(R+x)(-3) + (R-3x)(1)\right] = \frac{1}{3}\pi(-3R-3x+R-3x)$

$= \frac{1}{3}\pi(-2R-6x) = -\frac{2}{3}\pi(R+3x)$

$\therefore \left.\frac{d^2V}{dx^2}\right]_{x=\frac{R}{3}} = -\frac{2}{3}\pi(R+R) = -\frac{4}{3}\pi R < 0$ $\therefore$ V is max. when $x = \frac{R}{3}$.

Now, Volume of cone $= \frac{1}{3}\pi(R+x)^2(R-x)$

$= \frac{1}{3}\pi\left(R+\frac{R}{3}\right)^2\left(R-\frac{R}{3}\right)$

$= \frac{1}{3}\pi \times \frac{16}{9}R^2 \times \frac{2R}{3} = \frac{32}{81}\pi R^3$

$= \frac{8}{27}\left(\frac{4}{3}\pi R^3\right) = \frac{8}{27}$ (vol of sphere). H. Proved.

Ans 25.

Given: $P = 2x + 2y + \pi x = 10$

$\Rightarrow y = \frac{10-2x-\pi x}{2}$

To max.: Area (A)

$A = (2x)(y) + \frac{1}{2}\pi x^2$

$= 2x\left(\frac{10-2x-\pi x}{\cancel{2}}\right) + \frac{1}{2}\pi x^2 = 10x - 2x^2 - \pi x^2 + \frac{1}{2}\pi x^2$

$\Rightarrow A = 10x - 2x^2 - \frac{1}{2}\pi x^2$

$\frac{dA}{dx} = 10 - 4x - \pi x$, for max|min $\frac{dA}{dx} = 0 \Rightarrow 10 = 4x + \pi x$

$\Rightarrow x = \frac{10}{\pi+4}$ m

also, $\frac{d^2A}{dx^2} = -(4+\pi) < 0$ $\therefore$ A is max when $x = \frac{10}{\pi+4}$ m.

Now, $y = \frac{1}{2}(10-2x-\pi x) = \frac{1}{2}(10-(2+\pi).x) = \frac{1}{2}\left(10 - (2+\pi).\frac{10}{\pi+4}\right)$

$= 5\left(\frac{\pi+4-2-\pi}{\pi+4}\right) = \frac{10}{\pi+4}$ m

$\therefore$ Length $= \frac{20}{\pi+4}$ m and breadth $= \frac{10}{\pi+4}$ m Ans.

Ans 26. for 1-1 let $x_1, x_2 \in N$ (domain)

Let $f(x_1) = f(x_2)$

$\Rightarrow 4x_1^2 + 12x_1 + 15 = 4x_2^2 + 12x_2 + 15$

$\Rightarrow 4x_1^2 - 4x_2^2 = 12x_2 - 12x_1$

$\Rightarrow 4(x_1-x_2)(x_1+x_2) + 12(x_1-x_2) = 0$

$\Rightarrow 4(x_1-x_2)(x_1+x_2+3) = 0$

$\Rightarrow x_1 - x_2 = 0$ $\quad (\because x_1 + x_2 + 3 > 0$ as $x_1, x_2 \in N)$

$\Rightarrow x_1 = x_2$

$\Rightarrow f$ is 1-1.

Since, $f: N \to R_f$ $\Rightarrow$ Range = Co-domain (given).

$\Rightarrow f$ is onto.

Thus f is 1-1 and onto $\therefore f$ is invertible.

For f^{-1}. Let $y = f(x)$. $\Rightarrow x = f^{-1}(y)$ ——— ①

$\Rightarrow y = 4x^2 + 12x + 15$

$= (2x)^2 + 2(2x)(3) + (3)^2 - (3)^2 + 15$

$= (2x+3)^2 + 6$

$\Rightarrow (2x+3)^2 = y - 6$

$\Rightarrow 2x + 3 = \pm\sqrt{y-6}$

$\Rightarrow 2x + 3 = \sqrt{y-6}$ $\quad (\because 2x+3 > 0)$

$\Rightarrow x = \dfrac{\sqrt{y-6} - 3}{2}$ ——— ②

from ① & ②,

$f^{-1}(y) = \dfrac{\sqrt{y-6} - 3}{2}$

i.e. $f^{-1}(x) = \dfrac{\sqrt{x-6} - 3}{2}$ Ans.

Ans 21. (a) Given expression, LHS

$$= \cot^{-1}\left(\frac{\sqrt{\cos^2 x/2 + \sin^2 x/2 + 2\cos\frac{x}{2}\sin\frac{x}{2}} + \sqrt{\cos^2\frac{x}{2} + \sin^2\frac{x}{2} - 2\sin\frac{x}{2}\cos\frac{x}{2}}}{\sqrt{\cos^2\frac{x}{2} + \sin^2\frac{x}{2} + 2\cos\frac{x}{2}\sin\frac{x}{2}} - \sqrt{\cos^2\frac{x}{2} + \sin^2\frac{x}{2} - 2\sin\frac{x}{2}\cos\frac{x}{2}}}\right)$$

$$= \cot^{-1}\left(\frac{\cos\frac{x}{2} + \sin\frac{x}{2} + \cos\frac{x}{2} - \sin\frac{x}{2}}{\cos x/2 + \sin\frac{x}{2} - \cos\frac{x}{2} + \sin\frac{x}{2}}\right)$$

$$= \cot^{-1}\left(\frac{2\cos x/2}{2\sin x/2}\right) = \cot^{-1}\left(\cot\frac{x}{2}\right)$$

$= \dfrac{x}{2}$ = RHS. H. Proved.

(b) Given expression

$= \tan\frac{1}{2}\left[2\tan^{-1}x + 2\tan^{-1}y\right] = \tan(\tan^{-1}x + \tan^{-1}y)$

$= \tan\left(\tan^{-1}\left(\frac{x+y}{1-xy}\right)\right) = \frac{x+y}{1-xy}$ Ans

Ans 28. (a) $\frac{dx}{d\theta} = a(-\sin\theta + \theta\cos\theta + \sin\theta) = a\theta\cos\theta$

$\frac{dy}{d\theta} = a\left[\cos\theta - (\theta(-\sin\theta) + \cos\theta . 1)\right] = a\theta\sin\theta$

Now, $\frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{a\theta\sin\theta}{a\theta\cos\theta} = \underline{\tan\theta}$ Ans.

(b) f being polynomial fn. is cts in $[-4,2]$.

f being poly. fn. is diff. in $(-4,2)$ with $f'(x) = 2x+2$.

$f(-4) = (-4)^2 + 2(-4) - 8 = 0$

$f(2) = (2)^2 + 2(2) - 8 = 0 \Rightarrow f(-4) = f(2)$.

So, all conditions of Rolle's theorem are satisfied.

$\therefore$ there exist some $c \in (-4,2)$ such that $f'(c) = 0$

$\Rightarrow 2c + 2 = 0 \Rightarrow \underline{c = -1} \in (-4,2)$.

Hence verified.

Ans 29. $I = \int\left(\frac{\sqrt{\sin x}}{\sqrt{\cos x}} + \frac{\sqrt{\cos x}}{\sqrt{\sin x}}\right)dx = \int \frac{\sin x + \cos x}{\sqrt{\sin x \cos x}}\,dx$

$= \sqrt{2}\int \frac{\sin x + \cos x}{\sqrt{\sin 2x}}\,dx$

Put $\cos x - \sin x = t \Rightarrow (-\sin x - \cos x)\,dx = dt$

also, $\cos^2 x + \sin^2 x - 2\cos x \sin x = t^2$

$\Rightarrow 1 - \sin 2x = t^2 \Rightarrow \sin 2x = 1 - t^2$

$\therefore I = \sqrt{2}\int \frac{-dt}{\sqrt{1-t^2}} = -\sqrt{2}.\sin^{-1}t + c$

$= -\sqrt{2}\sin^{-1}(\cos x - \sin x) + c$ Ans

OR

Put $\dfrac{x^2+1}{(x-1)^2(x+3)} = \dfrac{A}{x+3} + \dfrac{B}{(x-1)} + \dfrac{C}{(x-1)^2}$

$= \dfrac{A(x-1)^2 + B(x-1)(x+3) + C(x+3)}{(x-1)^2(x+3)}$

$\Rightarrow x^2+1 = A(x-1)^2 + B(x-1)(x+3) + C(x+3)$

$= A(x^2-2x+1) + B(x^2+2x-3) + C(x+3)$

$= (A+B)x^2 + (-2A+2B+C).x + (A-3B+3C)$

Comparing, $A+B = 1$

$-2A+2B+C = 0$

$A-3B+3C = 1$

solving, $A = \frac{5}{8}$

$B = \frac{3}{8}$

$C = \frac{1}{2}$

$\underline{x=1}$

$2 = 4C \Rightarrow C = \frac{1}{2}$

$\underline{x=-3}$

$10 = 16A \Rightarrow A = \frac{5}{8}$

$B = 1-A$

$= 1 - \frac{5}{8}$

$\therefore I = \frac{5}{8}\int \frac{dx}{x+3} + \frac{3}{8}\int \frac{dx}{x-1} + \frac{1}{2}\int \frac{dx}{(x-1)^2}$

$= \frac{5}{8}\log|x+3| + \frac{3}{8}\log|x-1| - \frac{1}{2(x-1)} + c$ Ans.