

ASSIGNMENT CLASS XII APPLICATION OF DERIVATIVES

Questions based on Mean Value Theorem

1. Verify Rolle's Theorem for each of the following functions:

(a) $f(x) = x^2 - x - 6$ in $[-2, 3]$ (b) $f(x) = x^3 - 7x^2 + 16x - 12$ in $[2, 3]$

(c) $f(x) = (x-1)(x-2)^2$ in $[1, 2]$ (d) $f(x) = \sin x + \cos x$ in $\left[0, \frac{\pi}{2}\right]$

2. Discuss the applicability of the Rolle's Theorem for the following functions on the indicated intervals:

(a) $f(x) = x^{2/3}$ on $[-1, 1]$ (b) $f(x) = |x|$ on $[-1, 1]$

(c) $f(x) = [x]$ on $[-1, 1]$ (d) $f(x) = \begin{cases} x^2 + 1 & \text{when } 0 < x \leq 1 \\ 3 - x & \text{when } 1 < x \leq 2 \end{cases}$ on $[0, 2]$

3. Using Rolle's Theorem, find the points on the curve $y = 16 - x^2$, $x \in [-1, 1]$, where the tangent is parallel to x -axis.

4. Verify Lagrange's Mean Value Theorem for each of the following functions:

(a) $f(x) = \sqrt{x^2 - 1}$ in $[2, 4]$ (b) $f(x) = x^2 + x - 1$ in $[0, 4]$

(c) $f(x) = x^2 - 2x + 4$ on $[1, 5]$ (d) $f(x) = x^3 + x^2 - 6x$ on $[-1, 4]$

5. Find a point on the parabola $y = (x-3)^2$, where the tangent is parallel to chord joining $(3, 0)$ and $(4, 1)$.

6. Use Lagrange's Mean Value Theorem to determine a point P on the curve $f(x) = \sqrt{x-2}$ defined on $[2, 3]$, where the tangent is parallel to the chord joining the end points of the curve.

Questions based on Rate of Change of Quantities

7. Find the points on the curve $y^2 = 8x$ for which abscissa and ordinate change at the same rate.

8. A particle moves along the curve $y = \frac{2}{3}x^3 + 1$. Find the points on the curve at which y -coordinate is changing twice as fast as x -coordinate.

9. The volume of spherical balloon is increasing at the rate of $25 \text{ cm}^3 / \text{sec}$. Find the rate of change of its surface area at the instant when its radius is 5 cm.

10. The surface area of a spherical bubble is increasing at the rate of $2 \text{ cm}^2 / \text{sec}$. Find the rate at which the volume of the bubble is increasing at the instant its radius is 6 cm.

11. Water is leaking from a conical funnel at the rate of $5 \text{ cm}^3 / \text{sec}$. If the radius of the base of funnel is 5 cm and its altitude is 10 cm, find the rate at which water level is dropping when it is 2.5 cm from top.

12. Water is running into a conical vessel, 15 cm deep and 5 cm in radius, at the rate of $0.1 \text{ cm}^3 / \text{sec}$. When the water is 6 cm deep, find at what rate is:

(a) water level rising? (b) water surface area increasing? (c) wetted surface of vessel increasing?

Questions based on Increasing and Decreasing Functions

13. Show that the function $f(x) = x^3 - 6x^2 + 12x - 18$ is an increasing function on R .

14. Show that the function $f(x) = \cos^2 x$ is strictly decreasing on $\left(0, \frac{\pi}{2}\right)$.

15. Find the intervals on which the following functions are (i) increasing (ii) decreasing:

(a) $f(x) = 2x^3 - 15x^2 + 36x + 6$ (b) $f(x) = 5 + 36x + 3x^2 - 2x^3$

(c) $f(x) = \frac{4x^2 + 1}{x}$, $x \neq 0$ (d) $f(x) = \frac{x}{x^2 + 1}$ (e) $f(x) = (x+2)e^{-x}$

(f) $f(x) = (x-1)^3(x-2)^2$ (g) $f(x) = \frac{x}{2} + \frac{2}{x}$, $x \neq 0$ (h) $f(x) = \sin^4 x + \cos^4 x$ in $\left[0, \frac{\pi}{2}\right]$

Questions based on Tangent and Normal

16. Prove that the tangents to the curves $y=x^2-5x+6$ at the points $(2,0)$ and $(3,0)$ are at right angles.
17. Find points on the curve $\frac{x^2}{9}-\frac{y^2}{16}=1$ at which the tangents are parallel to (a) x-axis (b) y-axis.
18. Find the equation of the tangent to the curve $y=\sqrt{5x-3}-2$, which is parallel to the line $4x-2y+3=0$.
19. Find the equation of the normals to the curve $3x^2-y^2=8$, which is parallel to the line $x+3y=4$.
20. Find the equation of the tangent to the curve $x^2+3y=3$, which is parallel to the line $y-4x+5=0$.
21. At what points on the curve $x^2+y^2-2x-4y+1=0$, is the tangent parallel to the y-axis.
22. Show that the curves $xy=a^2$ and $x^2+y^2=2a^2$ touch each other.
23. Find the equation of the tangent to the curve $x=\theta+\sin\theta, y=1+\cos\theta$ at $\theta=\frac{\pi}{4}$.
24. Find the points on the curve $4x^2+9y^2=1$, where the tangents are perpendicular to the line $2y+x=0$.
25. If the tangent to the curve $y=x^3+ax+b$ at $(1,-6)$ is parallel to the line $x-y+5=0$, find a and b .
26. Find equation of the tangent to the curve $y=4x^3-3x+4$, which are perpendicular to $9y+x+3=0$.
27. (i) Show that the curves $2x=y^2$ and $2xy=k$ cut at right angles if $k^2=8$.
28. Find the equations of the tangent and the normal to the following curves at the indicated points:
- (a) $y=2x^2-3x-1$ at $(1,-2)$ (b) $y=x^2+4x+1$ at $x=3$
- (c) $y^2=\frac{x^3}{4-x}$ at $(2,-2)$ (d) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at $(a\cos\theta, b\sin\theta)$

Questions based on Approximations

29. Using differentials, find the approximate value of the following:
- (a) $\sqrt{0.037}$ (b) $\sqrt{0.48}$ (c) $\sqrt[3]{29}$ (d) $(33.1)^{1/5}$
30. If $y=x^4-12$ and if x changes from 2 to 1.99, what is the approximate change in the y ?
31. Find approximate change in volume V of a cube of side x meters caused by increasing the side by 2%.
32. If the radius of a sphere is measured as 9 cm with an error of 0.03 cm, find approximating error in calculating its volume.

Questions based on Maxima and Minima

33. Find the points of local maxima or local minima and the corresponding local maximum and minimum values of each of the following functions:
- (a) $f(x)=2x^3-21x^2+36x-20$ (b) $f(x)=x^4-62x^2+120x+9$
- (c) $f(x)=(x-1)(x+2)^2$ (d) $f(x)=\frac{-3}{4}x^4-8x^3-\frac{45}{2}x^2+105$
- (e) $f(x)=\sin 2x-x$, where $-\frac{\pi}{2}\leq x\leq\frac{\pi}{2}$ (f) $f(x)=2\cos x+x$, where $0<x<\pi$
34. Find the absolute maximum value and absolute minimum value of the following functions:
- (a) $f(x)=4x-\frac{x^2}{2}$ in $[-2, 4.5]$ (b) $f(x)=3x^4-8x^3+12x^2-48x+1$ in $[1, 4]$
35. Find the maximum and minimum values of the following functions:
- (a) $f(x)=-x+2\sin x$ on $[0, 2\pi]$ (b) $f(x)=\left(\sin x+\frac{1}{2}\cos x\right)$ on $0\leq x\leq\frac{\pi}{2}$
36. Show that $f(x)=\sin x(1+\cos x)$ is maximum at $x=\frac{\pi}{3}$ in the interval $[0, \pi]$.

Questions based on Maxima and Minima(Word Problems)

37. Show that of all the rectangles of the given area, the square has the smallest perimeter.
38. Find the point on the curve $y^2 = 4x$ which is nearest to the point $(2,1)$.
39. A rectangle is inscribed in a semicircle of radius r with one of its sides on the diameter of the semicircle. Find the dimensions of the rectangle, so that its area is maximum. Also find the maximum area.
40. A right circular cylinder is inscribed in a given cone. Show that the curved surface area of the cylinder is maximum when diameter of cylinder is equal to the radius of the base of cone.
41. An open tank with the square base and vertical sides is to be constructed from a metal sheet so as to hold a given quantity of water. Show that the cost of the material will be least when the depth of the tank is half of its width.
42. Show that a closed right circular cylinder of given surface area and maximum volume is such that its height is equal to the diameter of the base.
43. Show that the height of the right circular cylinder of maximum volume that can be inscribed in a given right circular cone of height h is $\frac{h}{3}$.
44. Find the largest possible area of a right angled triangle whose hypotenuse is 5 cm long.
45. A closed circular cylinder has a volume of 2156 cu. cm. What will be the radius of its base so that its total surface area is minimum?
46. Of all the rectangles each of which has a perimeter 40 m, find one which has maximum area. Find the maximum area also.
47. A wire of length 20 m is to be cut into two pieces. One of the pieces will be bent into shape of a square and the other into shape of an equilateral triangle. Where the wire should be cut so that the sum of the areas of the square and triangle is minimum?
48. An open box with a square base is to be made out of a given quantity of sheet of area a^2 . Show that the maximum volume of the box is $\frac{a^3}{6\sqrt{3}}$.
49. Show that the volume of the greatest cylinder which can be inscribed in a cone of height h and semi-vertical angle 30° is $\frac{4}{81}\pi h^3$.
50. Show that a right triangle of given hypotenuse has maximum area when it is an isosceles triangle.
51. A window is in the form of an rectangle above which there is a semi-circle. If the perimeter of the window is p cm, show that the window will allow the maximum possible light only when the radius of the semi-circle is $\frac{p}{\pi+4}$.
52. Show that the rectangle of maximum area that can be inscribed in a circle of radius r is a square of side $\sqrt{2}r$.
53. Two sides of a triangle have lengths a and b and the angle between them is θ . What value of θ will maximize the area of the triangle? Find the maximum area of the triangle also.
54. A rectangular window is surmounted by an equilateral triangle. Given that the perimeter is $16m$, find the width of the window so that the maximum amount of light may enter.
55. Show that a right circular cylinder which is open at top, and has a given surface area, will have greatest volume if its height is equal to the radius of its base.
56. Show that the maximum volume of the cylinder which can be inscribed in the sphere of radius $5\sqrt{3}cm$ is $500\pi cm^3$.
57. Show that the right circular cylinder of given volume open at the top has minimum total surface area, provided its height is equal to the radius of its base.
58. Prove that the surface area of the solid cuboid, of square base and given volume, is minimum when it is a cube.
59. Show that the height of the cone of maximum volume that can be inscribed in a sphere of radius $12cm$ is $16cm$.
60. A jet of an enemy is flying along the curve $y=x^2+2$. A soldier is placed at the point $(3,2)$. What is the nearest distance between the soldier and the jet?

ANSWERS (APPLICATION OF DERIVATIVES)

1. (a) $c = \frac{1}{2}$ (b) $c = \frac{8}{3}$ (c) $\frac{4}{3}$ (d) $c = \frac{\pi}{4}$ 2. Not applicable in any case 3. (0, 16)

4. (a) $c = \sqrt{6}$ (b) $c = 2$ (c) $c = 3$ (d) $c = \frac{\sqrt{31}-1}{3}$ 5. $\left(\frac{7}{2}, \frac{1}{4}\right)$ 6. $\left(\frac{9}{4}, \frac{1}{2}\right)$ 7. (2, 4)

8. $\left(1, \frac{5}{3}\right); \left(-1, \frac{1}{3}\right)$ 9. $10 \text{ cm}^2 / \text{sec}$ 10. $6 \text{ cm}^3 / \text{sec}$ 11. $\frac{-16}{45\pi} \text{ cm} / \text{sec}$ 12. (a) $\frac{1}{40\pi} \text{ cm} / \text{sec}$

(b) $\frac{1}{30} \text{ cm}^2 / \text{sec}$ (c) $\frac{\sqrt{10}}{30} \text{ cm}^2 / \text{sec}$ 15. (a) inc. on $(-\infty, 2] \cup [3, \infty)$, dec. on $[2, 3]$

(b) inc. on $[-2, 3]$, dec. on $(-\infty, -2] \cup [3, \infty)$ (c) inc. on $\left(-\infty, \frac{-1}{2}\right] \cup \left[\frac{1}{2}, \infty\right)$, dec. on $\left[\frac{-1}{2}, \frac{1}{2}\right]$

(d) inc. on $[-1, 1]$, dec. on $(-\infty, -1] \cup [1, \infty)$ (e) inc. on $[-\infty, -1]$, dec. on $[-1, \infty)$

(f) inc. on $\left(-\infty, \frac{8}{5}\right) \cup (2, \infty)$, dec. on $\left(\frac{8}{5}, 2\right)$ (g) inc. on $(-\infty, -2) \cup (2, \infty)$, dec. on $(-2, 0) \cup (0, 2)$

(h) inc. on $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$, dec. on $\left[0, \frac{\pi}{4}\right]$ 17. (a) No point (b) (3, 0) and (-3, 0)

18. $80x - 40y - 103 = 0$ 19. $x + 3y + 8 = 0$ 20. $4x - y + 13 = 0$ 21. (-1, 2) and (3, 2)

23. $y = (1 - \sqrt{2})x + \frac{(\sqrt{2}-1)\pi}{4} + 2$ 24. $\left(\frac{3}{2\sqrt{10}}, \frac{-1}{3\sqrt{10}}\right)$ and $\left(\frac{-3}{2\sqrt{10}}, \frac{1}{3\sqrt{10}}\right)$ 25. $a = -2, b = -5$

26. $9x - y - 3 = 0$ and $9x - y - 13 = 0$ 28. (a) $x - y - 3 = 0, x + y + 1 = 0$ (b) $10x - y - 8 = 0, x + 10y - 223 = 0$

(c) $2x + y - 2 = 0, x - 2y - 6 = 0$ (d) $\frac{x}{a} \cos \theta + \frac{y}{b} \sin \theta = 1, ax \sec \theta - by \csc \theta = a^2 - b^2$

29. (a) 0.1925 (b) 0.693 (c) 3.074 (d) 2.01375

30. -0.32 31. $0.06x^3 \text{ m}^3$ 32. $9.72\pi \text{ cm}^3$

33. (a) Local max. value is -3 at $x = 1$ and Local min. value is -128 at $x = 6$

(b) Local max. value is 68 at $x = 1$ and Local min. value is -1647 at $x = -6$ and -316 at $x = 5$

(c) Local max. value is 0 at $x = -2$ and Local min. value is -4 at $x = 0$

(d) Local max. value is 105 at $x = 0$ and $\frac{295}{4}$ at $x = -5$ and Local min. value is $\frac{231}{4}$ at $x = -3$

(e) Local max. value is $\frac{\sqrt{3}}{2} - \frac{\pi}{6}$ at $x = \frac{\pi}{6}$ and Local min. value is $-\left(\frac{\sqrt{3}}{2} + \frac{\pi}{6}\right)$ at $x = \frac{-\pi}{6}$

34. (a) Abs max. is 8 at $x = 4$, Abs. min. is -10 at $x = -2$

(b) Abs max. is 257 at $x = 4$, Abs. min. is -63 at $x = 2$

35. (a) max. value is $\left(-\frac{\pi}{3} + \sqrt{3}\right)$ at $x = \frac{\pi}{3}$ and min. value is $-\left(\frac{5\pi}{3} + \sqrt{3}\right)$ at $x = \frac{5\pi}{3}$

(b) max. value is $\frac{3}{4}$ at $x = \frac{\pi}{6}$ and min. value is $\frac{1}{2}$ at $x = \frac{\pi}{2}$ 38. (1, 2) 39. $\frac{r}{\sqrt{2}}$ units, $\sqrt{2}r$ units; r^2 units

44. $\frac{25}{4} \text{ cm}^2$ 45. 7 cm 46. square, 100 m^2 47. $\frac{20\sqrt{3}}{9+4\sqrt{3}}, \frac{60}{9+4\sqrt{3}}$ 53. $90^\circ, \frac{1}{2}ab$

54. $3.46m$ 60. $\sqrt{5}$