CLASS XII INTEGRALS CHAPTER 8

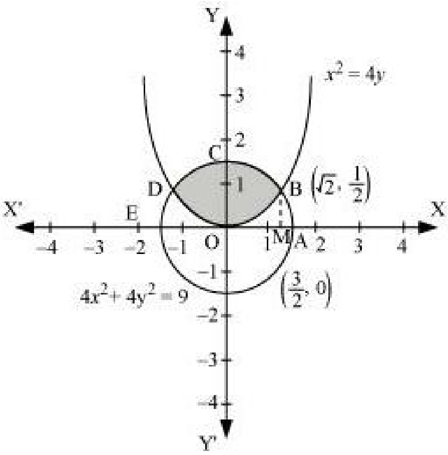
EX. 8.2 SOLUTIONS

Question 1:

Find the area of the circle $4x^2 + 4y^2 = 9$ which is interior to the parabola $x^2 = 4y$

ANS :

The required area is represented by the shaded area OBCDO.



Solving the given equation of circle, $4x^2 + 4y^2 = 9$, and parabola, $x^2 = 4y$, we obtain the point of intersection as B $\left(\sqrt{2}, \frac{1}{2}\right)$ and D $\left(-\sqrt{2}, \frac{1}{2}\right)$.

It can be observed that the required area is symmetrical about y-axis.

$\therefore$ Area OBCDO = 2 × Area OBCO

We draw BM perpendicular to OA.

Therefore, the coordinates of M are $\left(\sqrt{2}, 0\right)$.

Therefore, Area OBCO = Area OMBCO – Area OMBO

$$= \int_0^{\sqrt{2}} \sqrt{\frac{(9-4x^2)}{4}}\,dx - \int_0^{\sqrt{2}} \sqrt{\frac{x^2}{4}}\,dx$$
$$= \frac{1}{2}\int_0^{\sqrt{2}} \sqrt{9-4x^2}\,dx - \frac{1}{4}\int_0^{\sqrt{2}} x^2\,dx$$
$$= \frac{1}{4}\left[x\sqrt{9-4x^2} + \frac{9}{2}\sin^{-1}\frac{2x}{3}\right]_0^{\sqrt{2}} - \frac{1}{4}\left[\frac{x^3}{3}\right]_0^{\sqrt{2}}$$

$$=\frac{1}{4}\left[\sqrt{2}\sqrt{9-8}+\frac{9}{2}\sin^{-1}\frac{2\sqrt{2}}{3}\right]-\frac{1}{12}\left(\sqrt{2}\right)^3$$

$$=\frac{\sqrt{2}}{4}+\frac{9}{8}\sin^{-1}\frac{2\sqrt{2}}{3}-\frac{\sqrt{2}}{6}$$

$$=\frac{\sqrt{2}}{12}+\frac{9}{8}\sin^{-1}\frac{2\sqrt{2}}{3}$$

$$=\frac{1}{2}\left(\frac{\sqrt{2}}{6}+\frac{9}{4}\sin^{-1}\frac{2\sqrt{2}}{3}\right)$$

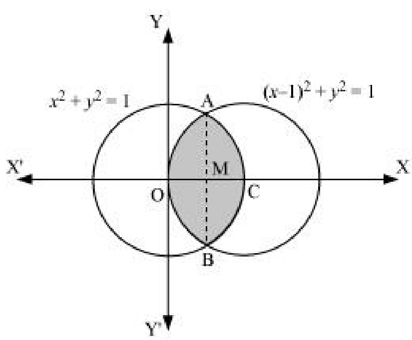
Therefore, the required area OBCDO is $\left(2\times\frac{1}{2}\left[\frac{\sqrt{2}}{6}+\frac{9}{4}\sin^{-1}\frac{2\sqrt{2}}{3}\right]\right)=\left[\frac{\sqrt{2}}{6}+\frac{9}{4}\sin^{-1}\frac{2\sqrt{2}}{3}\right]$ units

Question 2:

Find the area bounded by curves $(x-1)^2+y^2=1$ and $x^2+y^2=1$

ANS :

The area bounded by the curves, $(x-1)^2+y^2=1$ and $x^2+y^2=1$, is represented by the shaded area as



On solving the equations, $(x-1)^2+y^2=1$ and $x^2+y^2=1$, we obtain the point of intersection as $A\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$ and $B\left(\frac{1}{2},-\frac{\sqrt{3}}{2}\right)$.

It can be observed that the required area is symmetrical about x-axis.

$\therefore$ Area OBCAO = 2 × Area OCAO

We join AB, which intersects OC at M, such that AM is perpendicular to OC.

The coordinates of M are $\left(\frac{1}{2},0\right)$.

$\Rightarrow Area\ OCAO = Area\ OMAO + Area\ MCAM$

$$= \left[\int_0^{\frac{1}{2}} \sqrt{1-(x-1)^2}\,dx + \int_{\frac{1}{2}}^{1} \sqrt{1-x^2}\,dx\right]$$

$$= \left[\frac{x-1}{2}\sqrt{1-(x-1)^2} + \frac{1}{2}\sin^{-1}(x-1)\right]_0^{\frac{1}{2}} + \left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x\right]_{\frac{1}{2}}^{1}$$

$$= \left[-\frac{1}{4}\sqrt{1-\left(-\frac{1}{2}\right)^2} + \frac{1}{2}\sin^{-1}\left(\frac{1}{2}-1\right) - \frac{1}{2}\sin^{-1}(-1)\right] +$$

$$\left[\frac{1}{2}\sin^{-1}(1) - \frac{1}{4}\sqrt{1-\left(\frac{1}{2}\right)^2} - \frac{1}{2}\sin^{-1}\left(\frac{1}{2}\right)\right]$$

$$= \left[-\frac{\sqrt{3}}{8} + \frac{1}{2}\left(-\frac{\pi}{6}\right) - \frac{1}{2}\left(-\frac{\pi}{2}\right)\right] + \left[\frac{1}{2}\left(\frac{\pi}{2}\right) - \frac{\sqrt{3}}{8} - \frac{1}{2}\left(\frac{\pi}{6}\right)\right]$$

$$= \left[-\frac{\sqrt{3}}{4} - \frac{\pi}{12} + \frac{\pi}{4} + \frac{\pi}{4} - \frac{\pi}{12}\right]$$

$$= \left[-\frac{\sqrt{3}}{4} - \frac{\pi}{6} + \frac{\pi}{2}\right]$$

$$= \left[\frac{2\pi}{6} - \frac{\sqrt{3}}{4}\right]$$

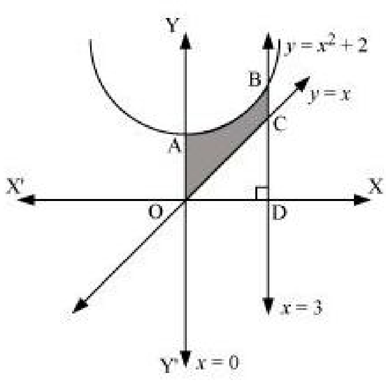
Therefore, required area OBCAO $= 2 \times \left(\frac{2\pi}{6} - \frac{\sqrt{3}}{4}\right) = \left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)$ units

Question 3:

Find the area of the region bounded by the curves $y = x^2 + 2$, $y = x$, $x = 0$ and $x = 3$

ANS :

The area bounded by the curves, $y = x^2 + 2$, $y = x$, $x = 0$, and $x = 3$, is represented by the shaded area OCBAO as



Then, Area OCBAO = Area ODBAO – Area ODCO

$$= \int_0^3 \left(x^2+2\right)dx - \int_0^3 x\,dx$$

$$= \left[\frac{x^3}{3}+2x\right]_0^3 - \left[\frac{x^2}{2}\right]_0^3$$

$$= [9+6] - \left[\frac{9}{2}\right]$$

$$= 15 - \frac{9}{2}$$

$$= \frac{21}{2} \text{ units}$$

Question 4:

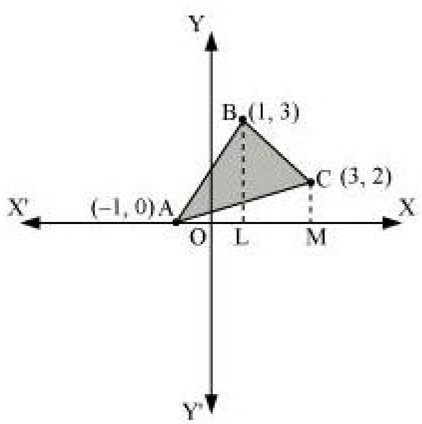
Using integration finds the area of the region bounded by the triangle whose vertices are (–1, 0), (1, 3) and (3, 2).

ANS :

BL and CM are drawn perpendicular to x-axis.

It can be observed in the following figure that,

Area (ΔACB) = Area (ALBA) + Area (BLMCB) – Area (AMCA) … (1)



Equation of line segment AB is

$$y-0=\frac{3-0}{1+1}(x+1)$$

$$y=\frac{3}{2}(x+1)$$

$$\therefore \text{Area}(\text{ALBA})=\int_{-1}^{1}\frac{3}{2}(x+1)dx=\frac{3}{2}\left[\frac{x^2}{2}+x\right]_{-1}^{1}=\frac{3}{2}\left[\frac{1}{2}+1-\frac{1}{2}+1\right]=3 \text{ units}$$

Equation of line segment BC is

$$y-3=\frac{2-3}{3-1}(x-1)$$

$$y=\frac{1}{2}(-x+7)$$

$$\therefore \text{Area}(\text{BLMCB})=\int_{1}^{3}\frac{1}{2}(-x+7)dx=\frac{1}{2}\left[-\frac{x^2}{2}+7x\right]_{1}^{3}=\frac{1}{2}\left[-\frac{9}{2}+21+\frac{1}{2}-7\right]=5 \text{ units}$$

Equation of line segment AC is

$$y-0=\frac{2-0}{3+1}(x+1)$$

$$y=\frac{1}{2}(x+1)$$

$$\therefore \text{Area}(\text{AMCA})=\frac{1}{2}\int_{-1}^{3}(x+1)dx=\frac{1}{2}\left[\frac{x^2}{2}+x\right]_{-1}^{3}=\frac{1}{2}\left[\frac{9}{2}+3-\frac{1}{2}+1\right]=4 \text{ units}$$

Therefore, from equation (1), we obtain

Area (ΔABC) = (3 + 5 – 4) = 4 units

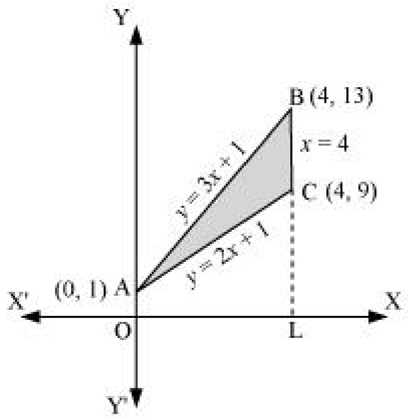
Question 5:

Using integration find the area of the triangular region whose sides have the equations $y = 2x + 1$, $y = 3x + 1$ and $x = 4$.

ANS :

The equations of sides of the triangle are $y = 2x + 1$, $y = 3x + 1$, and $x = 4$.

On solving these equations, we obtain the vertices of triangle as A(0, 1), B(4, 13), and C (4, 9).



It can be observed that,

Area (ΔACB) = Area (OLBAO) –Area (OLCAO)

$$= \int_0^4 (3x+1)\,dx - \int_0^4 (2x+1)\,dx$$

$$= \left[\frac{3x^2}{2}+x\right]_0^4 - \left[\frac{2x^2}{2}+x\right]_0^4$$

$$= (24+4)-(16+4)$$

$$= 28-20$$

$$= 8 \text{ units}$$

Question 6:

Smaller area enclosed by the circle $x^2 + y^2 = 4$ and the line $x + y = 2$ is

A. 2 (π – 2)

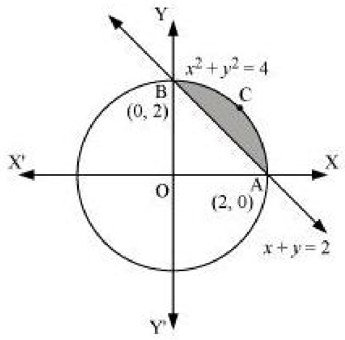
B. π – 2

C. 2π – 1

D. 2 (π + 2)

ANS :

The smaller area enclosed by the circle, $x^2 + y^2 = 4$, and the line, $x + y = 2$, is represented by the shaded area ACBA as



It can be observed that,

Area ACBA = Area OACBO – Area (ΔOAB)

$$= \int_0^2 \sqrt{4 - x^2}\, dx - \int_0^2 (2 - x)\, dx$$

$$= \left[\frac{x}{2}\sqrt{4 - x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2 - \left[2x - \frac{x^2}{2}\right]_0^2$$

$$= \left[2 \cdot \frac{\pi}{2}\right] - [4 - 2]$$

$$= (\pi - 2) \text{ units}$$

Thus, the correct answer is B.

Question 7:

Area lying between the curve $y^2 = 4x$ and $y = 2x$ is

A. $\frac{2}{3}$

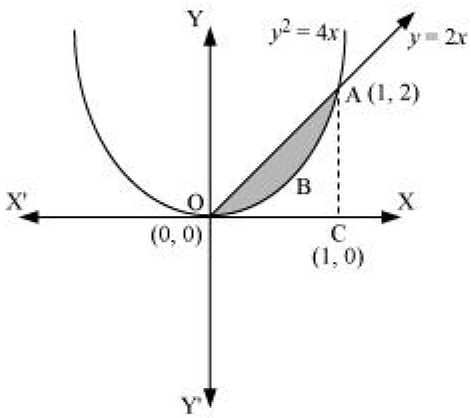
B. $\frac{1}{3}$

C. $\frac{1}{4}$

D. $\frac{3}{4}$

ANS :

The area lying between the curve, $y^2 = 4x$ and $y = 2x$, is represented by the shaded area OBAO as



The points of intersection of these curves are O (0, 0) and A (1, 2).

We draw AC perpendicular to x-axis such that the coordinates of C are (1, 0).

$\therefore$ Area OBAO = Area (ΔOCA) – Area (OCABO)

$$= \int_0^1 2x\,dx - \int_0^1 2\sqrt{x}\,dx$$

$$= 2\left[\frac{x^2}{2}\right]_0^1 - 2\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^1$$

$$= \left|1 - \frac{4}{3}\right|$$

$$= \left|-\frac{1}{3}\right|$$

$$= \frac{1}{3} \text{ units}$$