

CLASS XII INTEGRALS CHAPTER 7

EX. 7.8 SOLUTIONS

Question 1:

$$\int_a^b x dx$$

ANS :

It is known that,

$$\int_a^b f(x) dx = (b-a) \lim_{n \rightarrow \infty} \frac{1}{n} [f(a) + f(a+h) + \dots + f(a+(n-1)h)], \text{ where } h = \frac{b-a}{n}$$

Here, $a = a$, $b = b$, and $f(x) = x$

$$\begin{aligned} \therefore \int_a^b x dx &= (b-a) \lim_{n \rightarrow \infty} \frac{1}{n} [a + (a+h) \dots (a+2h) \dots a + (n-1)h] \\ &= (b-a) \lim_{n \rightarrow \infty} \frac{1}{n} \left[\underbrace{(a+a+a+\dots+a)}_{n \text{ times}} + (h+2h+3h+\dots+(n-1)h) \right] \\ &= (b-a) \lim_{n \rightarrow \infty} \frac{1}{n} [na + h(1+2+3+\dots+(n-1))] \\ &= (b-a) \lim_{n \rightarrow \infty} \frac{1}{n} \left[na + h \left\{ \frac{(n-1)(n)}{2} \right\} \right] \\ &= (b-a) \lim_{n \rightarrow \infty} \frac{1}{n} \left[na + \frac{n(n-1)h}{2} \right] \\ &= (b-a) \lim_{n \rightarrow \infty} \frac{n}{n} \left[a + \frac{(n-1)h}{2} \right] \\ &= (b-a) \lim_{n \rightarrow \infty} \left[a + \frac{(n-1)h}{2} \right] \\ &= (b-a) \lim_{n \rightarrow \infty} \left[a + \frac{(n-1)(b-a)}{2n} \right] \\ &= (b-a) \lim_{n \rightarrow \infty} \left[a + \frac{\left(1 - \frac{1}{n}\right)(b-a)}{2} \right] \\ &= (b-a) \left[a + \frac{(b-a)}{2} \right] \\ &= (b-a) \left[\frac{2a+b-a}{2} \right] \\ &= \frac{(b-a)(b+a)}{2} \\ &= \frac{1}{2}(b^2 - a^2) \end{aligned}$$

Question 2:

$$\int_0^5 (x+1) dx$$

ANS :

$$\text{Let } I = \int_0^5 (x+1) dx$$

It is known that,

$$\int_a^b f(x) dx = (b-a) \lim_{n \rightarrow \infty} \frac{1}{n} [f(a) + f(a+h) \dots f(a+(n-1)h)], \text{ where } h = \frac{b-a}{n}$$

Here, $a = 0$, $b = 5$, and $f(x) = (x+1)$

$$\Rightarrow h = \frac{5-0}{n} = \frac{5}{n}$$

$$\begin{aligned} \therefore \int_0^5 (x+1) dx &= (5-0) \lim_{n \rightarrow \infty} \frac{1}{n} \left[f(0) + f\left(\frac{5}{n}\right) + \dots + f\left((n-1)\frac{5}{n}\right) \right] \\ &= 5 \lim_{n \rightarrow \infty} \frac{1}{n} \left[1 + \left(\frac{5}{n} + 1\right) + \dots + \left\{ 1 + \left(\frac{5(n-1)}{n}\right) \right\} \right] \\ &= 5 \lim_{n \rightarrow \infty} \frac{1}{n} \left[\left(1 + 1 + 1 \dots 1\right) + \left[\frac{5}{n} + 2 \cdot \frac{5}{n} + 3 \cdot \frac{5}{n} + \dots + (n-1) \frac{5}{n} \right] \right] \\ &= 5 \lim_{n \rightarrow \infty} \frac{1}{n} \left[n + \frac{5}{n} \{1 + 2 + 3 \dots (n-1)\} \right] \\ &= 5 \lim_{n \rightarrow \infty} \frac{1}{n} \left[n + \frac{5}{n} \cdot \frac{(n-1)n}{2} \right] \\ &= 5 \lim_{n \rightarrow \infty} \frac{1}{n} \left[n + \frac{5(n-1)}{2} \right] \\ &= 5 \lim_{n \rightarrow \infty} \left[1 + \frac{5}{2} \left(1 - \frac{1}{n}\right) \right] \end{aligned}$$

$$= 5 \left[1 + \frac{5}{2} \right]$$

$$= 5 \left[\frac{7}{2} \right]$$

$$= \frac{35}{2}$$

Question 3:

$$\int_2^3 x^2 dx$$

ANS :

$$\int_a^b f(x) dx = (b-a) \lim_{n \rightarrow \infty} \frac{1}{n} [f(a) + f(a+h) + f(a+2h) \dots f\{a+(n-1)h\}], \text{ where } h = \frac{b-a}{n}$$

Here, $a = 2$, $b = 3$, and $f(x) = x^2$

$$\Rightarrow h = \frac{3-2}{n} = \frac{1}{n}$$

$$\begin{aligned} \therefore \int_2^3 x^2 dx &= (3-2) \lim_{n \rightarrow \infty} \frac{1}{n} \left[f(2) + f\left(2 + \frac{1}{n}\right) + f\left(2 + \frac{2}{n}\right) \dots f\left\{2 + (n-1)\frac{1}{n}\right\} \right] \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \left[(2)^2 + \left(2 + \frac{1}{n}\right)^2 + \left(2 + \frac{2}{n}\right)^2 + \dots \left(2 + \frac{(n-1)}{n}\right)^2 \right] \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \left[2^2 + \left\{ 2^2 + \left(\frac{1}{n}\right)^2 + 2 \cdot 2 \cdot \frac{1}{n} \right\} + \dots + \left\{ (2)^2 + \frac{(n-1)^2}{n^2} + 2 \cdot 2 \cdot \frac{(n-1)}{n} \right\} \right] \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \left[\left(2^2 + \dots + 2^2 \right) + \left\{ \left(\frac{1}{n}\right)^2 + \left(\frac{2}{n}\right)^2 + \dots + \left(\frac{n-1}{n}\right)^2 \right\} + 2 \cdot 2 \cdot \left\{ \frac{1}{n} + \frac{2}{n} + \frac{3}{n} + \dots + \frac{(n-1)}{n} \right\} \right] \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \left[4n + \frac{1}{n^2} \{ 1^2 + 2^2 + 3^2 \dots + (n-1)^2 \} + \frac{4}{n} \{ 1 + 2 + \dots + (n-1) \} \right] \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \left[4n + \frac{1}{n^2} \left\{ \frac{n(n-1)(2n-1)}{6} \right\} + \frac{4}{n} \left\{ \frac{n(n-1)}{2} \right\} \right] \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \left[4n + \frac{n \left(1 - \frac{1}{n} \right) \left(2 - \frac{1}{n} \right)}{6} + \frac{4n-4}{2} \right] \\ &= \lim_{n \rightarrow \infty} \left[4 + \frac{1}{6} \left(1 - \frac{1}{n} \right) \left(2 - \frac{1}{n} \right) + 2 - \frac{2}{n} \right] \\ &= 4 + \frac{2}{6} + 2 \\ &= \frac{19}{3} \end{aligned}$$

Question 4:

$$\int (x^2 - x) dx$$

ANS :

$$\begin{aligned}\text{Let } I &= \int_1^4 (x^2 - x) dx \\ &= \int_1^4 x^2 dx - \int_1^4 x dx\end{aligned}$$

$$\text{Let } I = I_1 - I_2, \text{ where } I_1 = \int_1^4 x^2 dx \text{ and } I_2 = \int_1^4 x dx \quad \dots(1)$$

It is known that,

$$\int_a^b f(x) dx = (b-a) \lim_{n \rightarrow \infty} \frac{1}{n} [f(a) + f(a+h) + f(a+(n-1)h)], \text{ where } h = \frac{b-a}{n}$$

$$\text{For } I_1 = \int_1^4 x^2 dx,$$

$$a = 1, b = 4, \text{ and } f(x) = x^2$$

$$\therefore h = \frac{4-1}{n} = \frac{3}{n}$$

$$\begin{aligned}I_1 &= \int_1^4 x^2 dx = (4-1) \lim_{n \rightarrow \infty} \frac{1}{n} [f(1) + f(1+h) + \dots + f(1+(n-1)h)] \\ &= 3 \lim_{n \rightarrow \infty} \frac{1}{n} \left[1^2 + \left(1 + \frac{3}{n}\right)^2 + \left(1 + 2 \cdot \frac{3}{n}\right)^2 + \dots + \left(1 + \frac{(n-1)3}{n}\right)^2 \right] \\ &= 3 \lim_{n \rightarrow \infty} \frac{1}{n} \left[1^2 + \left\{ 1^2 + \left(\frac{3}{n}\right)^2 + 2 \cdot \frac{3}{n} \right\} + \dots + \left\{ 1^2 + \left(\frac{(n-1)3}{n}\right)^2 + \frac{2 \cdot (n-1) \cdot 3}{n} \right\} \right] \\ &= 3 \lim_{n \rightarrow \infty} \frac{1}{n} \left[\left(1^2 + \dots + 1^2\right) + \left(\frac{3}{n}\right)^2 \left\{ 1^2 + 2^2 + \dots + (n-1)^2 \right\} + 2 \cdot \frac{3}{n} \left\{ 1 + 2 + \dots + (n-1) \right\} \right] \\ &= 3 \lim_{n \rightarrow \infty} \frac{1}{n} \left[n + \frac{9}{n^2} \left\{ \frac{(n-1)(n)(2n-1)}{6} \right\} + \frac{6}{n} \left\{ \frac{(n-1)(n)}{2} \right\} \right] \\ &= 3 \lim_{n \rightarrow \infty} \frac{1}{n} \left[n + \frac{9n}{6} \left(1 - \frac{1}{n}\right) \left(2 - \frac{1}{n}\right) + \frac{6n-6}{2} \right]\end{aligned}$$

$$\begin{aligned}
&= 3 \lim_{n \rightarrow \infty} \left[1 + \frac{9}{6} \left(1 - \frac{1}{n} \right) \left(2 - \frac{1}{n} \right) + 3 - \frac{3}{n} \right] \\
&= 3[1 + 3 + 3] \\
&= 3[7]
\end{aligned}$$

$$I_1 = 21 \quad \dots(2)$$

For $I_2 = \int_1^4 x dx$,

$a = 1, b = 4$, and $f(x) = x$

$$\Rightarrow h = \frac{4-1}{n} = \frac{3}{n}$$

$$\therefore I_2 = (4-1) \lim_{n \rightarrow \infty} \frac{1}{n} [f(1) + f(1+h) + \dots + f(a+(n-1)h)]$$

$$= 3 \lim_{n \rightarrow \infty} \frac{1}{n} [1 + (1+h) + \dots + (1+(n-1)h)]$$

$$= 3 \lim_{n \rightarrow \infty} \frac{1}{n} \left[1 + \left(1 + \frac{3}{n} \right) + \dots + \left\{ 1 + (n-1) \frac{3}{n} \right\} \right]$$

$$= 3 \lim_{n \rightarrow \infty} \frac{1}{n} \left[\underbrace{(1+1+\dots+1)}_{n \text{ times}} + \frac{3}{n} (1+2+\dots+(n-1)) \right]$$

$$= 3 \lim_{n \rightarrow \infty} \frac{1}{n} \left[n + \frac{3}{n} \left\{ \frac{(n-1)n}{2} \right\} \right]$$

$$= 3 \lim_{n \rightarrow \infty} \frac{1}{n} \left[1 + \frac{3}{2} \left(1 - \frac{1}{n} \right) \right]$$

$$= 3 \left[1 + \frac{3}{2} \right]$$

$$= 3 \left[\frac{5}{2} \right]$$

$$I_2 = \frac{15}{2} \quad \dots(3)$$

From equations (2) and (3), we obtain

$$I = I_1 + I_2 = 21 - \frac{15}{2} = \frac{27}{2}$$

Question 5:

$$\int_{-1}^1 e^x dx$$

ANS :

$$\text{Let } I = \int_{-1}^1 e^x dx \quad \dots(1)$$

It is known that,

$$\int_a^b f(x) dx = (b-a) \lim_{n \rightarrow \infty} \frac{1}{n} [f(a) + f(a+h) \dots f(a+(n-1)h)], \text{ where } h = \frac{b-a}{n}$$

Here, $a = -1$, $b = 1$, and $f(x) = e^x$

$$\therefore h = \frac{1+1}{n} = \frac{2}{n}$$

$$\therefore I = (1+1) \lim_{n \rightarrow \infty} \frac{1}{n} \left[f(-1) + f\left(-1 + \frac{2}{n}\right) + f\left(-1 + 2 \cdot \frac{2}{n}\right) + \dots + f\left(-1 + \frac{(n-1)2}{n}\right) \right]$$

$$= 2 \lim_{n \rightarrow \infty} \frac{1}{n} \left[e^{-1} + e^{\left(-1 + \frac{2}{n}\right)} + e^{\left(-1 + 2 \cdot \frac{2}{n}\right)} + \dots + e^{\left(-1 + \frac{(n-1)2}{n}\right)} \right]$$

$$= 2 \lim_{n \rightarrow \infty} \frac{1}{n} \left[e^{-1} \left\{ 1 + e^{\frac{2}{n}} + e^{\frac{4}{n}} + e^{\frac{6}{n}} + e^{\frac{(n-1)2}{n}} \right\} \right]$$

$$= 2 \lim_{n \rightarrow \infty} \frac{e^{-1}}{n} \left[\frac{e^{\frac{2n}{n}} - 1}{e^{\frac{2}{n}} - 1} \right]$$

$$= e^{-1} \times 2 \lim_{n \rightarrow \infty} \frac{1}{n} \left[\frac{e^2 - 1}{e^{\frac{2}{n}} - 1} \right]$$

$$= \frac{e^{-1} \times 2(e^2 - 1)}{\lim_{\frac{2}{n} \rightarrow 0} \left(\frac{e^{\frac{2}{n}} - 1}{\frac{2}{n}} \right) \times 2}$$

$$= e^{-1} \left[\frac{2(e^2 - 1)}{2} \right]$$

$$\left[\lim_{h \rightarrow 0} \left(\frac{e^h - 1}{h} \right) = 1 \right]$$

$$= \frac{e^2 - 1}{e}$$

$$= \left(e - \frac{1}{e} \right)$$

Question 6:

$$\int_0^1 (x + e^{2x}) dx$$

ANS :

$$\int_a^b f(x) dx = (b-a) \lim_{n \rightarrow \infty} \frac{1}{n} [f(a) + f(a+h) + \dots + f(a+(n-1)h)], \text{ where } h = \frac{b-a}{n}$$

Here, $a = 0$, $b = 4$, and $f(x) = x + e^{2x}$

$$\therefore h = \frac{4-0}{n} = \frac{4}{n}$$

$$\begin{aligned} \Rightarrow \int_0^4 (x + e^{2x}) dx &= (4-0) \lim_{n \rightarrow \infty} \frac{1}{n} [f(0) + f(h) + f(2h) + \dots + f((n-1)h)] \\ &= 4 \lim_{n \rightarrow \infty} \frac{1}{n} [(0 + e^0) + (h + e^{2h}) + (2h + e^{2 \cdot 2h}) + \dots + \{(n-1)h + e^{2(n-1)h}\}] \\ &= 4 \lim_{n \rightarrow \infty} \frac{1}{n} [1 + (h + e^{2h}) + (2h + e^{4h}) + \dots + \{(n-1)h + e^{2(n-1)h}\}] \\ &= 4 \lim_{n \rightarrow \infty} \frac{1}{n} [\{h + 2h + 3h + \dots + (n-1)h\} + \{1 + e^{2h} + e^{4h} + \dots + e^{2(n-1)h}\}] \\ &= 4 \lim_{n \rightarrow \infty} \frac{1}{n} \left[h \{1 + 2 + \dots + (n-1)\} + \left(\frac{e^{2hn} - 1}{e^{2h} - 1} \right) \right] \\ &= 4 \lim_{n \rightarrow \infty} \frac{1}{n} \left[\frac{(h(n-1)n)}{2} + \left(\frac{e^{2hn} - 1}{e^{2h} - 1} \right) \right] \\ &= 4 \lim_{n \rightarrow \infty} \frac{1}{n} \left[\frac{4}{n} \cdot \frac{(n-1)n}{2} + \left(\frac{e^8 - 1}{e^{\frac{8}{n}} - 1} \right) \right] \\ &= 4(2) + 4 \lim_{n \rightarrow \infty} \left(\frac{e^8 - 1}{\left(\frac{e^{\frac{8}{n}} - 1}{\frac{8}{n}} \right)} \right) \\ &= 8 + \frac{4 \cdot (e^8 - 1)}{8} \qquad \left(\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1 \right) \\ &= 8 + \frac{e^8 - 1}{2} \end{aligned}$$

$$= \frac{15 + e^8}{2}$$