

CLASS XII THREE DIMENSIONAL GEOMETRY CHAPTER 11

EX. 11.3 SOLUTIONS

Question 1:

In each of the following cases, determine the direction cosines of the normal to the plane and the distance from the origin.

(a) $z = 2$ (b) $x + y + z = 1$

(c) $2x + 3y - z = 5$ (d) $5y + 8 = 0$

ANS :

(a) The equation of the plane is $z = 2$ or $0x + 0y + z = 2$... (1)

The direction ratios of normal are 0, 0, and 1.

$$\therefore \sqrt{0^2 + 0^2 + 1^2} = 1$$

Dividing both sides of equation (1) by 1, we obtain

$$0.x + 0.y + 1.z = 2$$

This is of the form $lx + my + nz = d$, where l, m, n are the direction cosines of normal to the plane and d is the distance of the perpendicular drawn from the origin.

Therefore, the direction cosines are 0, 0, and 1 and the distance of the plane from the origin is 2 units.

(b) $x + y + z = 1$... (1)

The direction ratios of normal are 1, 1, and 1.

$$\therefore \sqrt{(1)^2 + (1)^2 + (1)^2} = \sqrt{3}$$

Dividing both sides of equation (1) by $\sqrt{3}$, we obtain

$$\frac{1}{\sqrt{3}}x + \frac{1}{\sqrt{3}}y + \frac{1}{\sqrt{3}}z = \frac{1}{\sqrt{3}} \quad \dots(2)$$

This equation is of the form $lx + my + nz = d$, where l, m, n are the direction cosines of normal to the plane and d is the distance of normal from the origin.

Therefore, the direction cosines of the normal are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$, and $\frac{1}{\sqrt{3}}$ and the distance of normal from the origin is $\frac{1}{\sqrt{3}}$ units.

(c) $2x + 3y - z = 5$... (1)

The direction ratios of normal are 2, 3, and -1.

$$\therefore \sqrt{(2)^2 + (3)^2 + (-1)^2} = \sqrt{14}$$

Dividing both sides of equation (1) by $\sqrt{14}$, we obtain

$$\frac{2}{\sqrt{14}}x + \frac{3}{\sqrt{14}}y - \frac{1}{\sqrt{14}}z = \frac{5}{\sqrt{14}}$$

This equation is of the form $lx + my + nz = d$, where l, m, n are the direction cosines of normal to the plane and d is the distance of normal from the origin.

Therefore, the direction cosines of the normal to the plane are $\frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}$, and $\frac{-1}{\sqrt{14}}$ and the distance of normal from the origin is $\frac{5}{\sqrt{14}}$ units.

$$(d) 5y + 8 = 0$$

$$\Rightarrow 0x - 5y + 0z = 8 \dots (1)$$

The direction ratios of normal are 0, -5, and 0.

$$\therefore \sqrt{0+(-5)^2+0} = 5$$

Dividing both sides of equation (1) by 5, we obtain

$$-y = \frac{8}{5}$$

This equation is of the form $lx + my + nz = d$, where l, m, n are the direction cosines of normal to the plane and d is the distance of normal from the origin.

Therefore, the direction cosines of the normal to the plane are 0, -1, and 0 and the distance of normal from the origin is $\frac{8}{5}$ units.

Question 2:

Find the vector equation of a plane which is at a distance of 7 units from the origin and normal to the vector $3\hat{i} + 5\hat{j} - 6\hat{k}$.

ANS :

The normal vector is, $\vec{n} = 3\hat{i} + 5\hat{j} - 6\hat{k}$

$$\therefore \hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{3\hat{i} + 5\hat{j} - 6\hat{k}}{\sqrt{(3)^2 + (5)^2 + (6)^2}} = \frac{3\hat{i} + 5\hat{j} - 6\hat{k}}{\sqrt{70}}$$

It is known that the equation of the plane with position vector \vec{r} is given by, $\vec{r} \cdot \hat{n} = d$

$$\Rightarrow \vec{r} \cdot \left(\frac{3\hat{i} + 5\hat{j} - 6\hat{k}}{\sqrt{70}} \right) = 7$$

This is the vector equation of the required plane.

Question 3:

Find the Cartesian equation of the following planes:

$$(a) \vec{r} \cdot (\hat{i} + \hat{j} - \hat{k}) = 2 \quad (b) \vec{r} \cdot (2\hat{i} + 3\hat{j} - 4\hat{k}) = 1$$

$$(c) \vec{r} \cdot [(s-2t)\hat{i} + (3-t)\hat{j} + (2s+t)\hat{k}] = 15$$

ANS :

(a) It is given that equation of the plane is

$$\vec{r} \cdot (\hat{i} + \hat{j} - \hat{k}) = 2 \quad \dots(1)$$

For any arbitrary point P (x, y, z) on the plane, position vector \vec{r} is given by, $\vec{r} = x\hat{i} + y\hat{j} - z\hat{k}$

Substituting the value of \vec{r} in equation (1), we obtain

$$\begin{aligned} (x\hat{i} + y\hat{j} - z\hat{k}) \cdot (\hat{i} + \hat{j} - \hat{k}) &= 2 \\ \Rightarrow x + y - z &= 2 \end{aligned}$$

This is the Cartesian equation of the plane.

$$(b) \vec{r} \cdot (2\hat{i} + 3\hat{j} - 4\hat{k}) = 1 \quad \dots(1)$$

For any arbitrary point P (x, y, z) on the plane, position vector \vec{r} is given by, $\vec{r} = x\hat{i} + y\hat{j} - z\hat{k}$

Substituting the value of \vec{r} in equation (1), we obtain

$$\begin{aligned} (x\hat{i} + y\hat{j} + z\hat{k}) \cdot (2\hat{i} + 3\hat{j} - 4\hat{k}) &= 1 \\ \Rightarrow 2x + 3y - 4z &= 1 \end{aligned}$$

This is the Cartesian equation of the plane.

$$(c) \vec{r} \cdot [(s-2t)\hat{i} + (3-t)\hat{j} + (2s+t)\hat{k}] = 15 \quad \dots(1)$$

For any arbitrary point P (x, y, z) on the plane, position vector \vec{r} is given by, $\vec{r} = x\hat{i} + y\hat{j} - z\hat{k}$

Substituting the value of \vec{r} in equation (1), we obtain

$$\begin{aligned} (x\hat{i} + y\hat{j} - z\hat{k}) \cdot [(s-2t)\hat{i} + (3-t)\hat{j} + (2s+t)\hat{k}] &= 15 \\ \Rightarrow (s-2t)x + (3-t)y + (2s+t)z &= 15 \end{aligned}$$

This is the Cartesian equation of the given plane.

Question 4:

In the following cases, find the coordinates of the foot of the perpendicular drawn from the origin.

(a) $2x + 3y + 4z - 12 = 0$ (b) $3y + 4z - 6 = 0$

(c) $x + y + z = 1$ (d) $5y + 8 = 0$

ANS :

(a) Let the coordinates of the foot of perpendicular P from the origin to the plane be (x_1, y_1, z_1) .

$$2x + 3y + 4z - 12 = 0$$

$$\Rightarrow 2x + 3y + 4z = 12 \dots (1)$$

The direction ratios of normal are 2, 3, and 4.

$$\therefore \sqrt{(2)^2 + (3)^2 + (4)^2} = \sqrt{29}$$

Dividing both sides of equation (1) by $\sqrt{29}$, we obtain

$$\frac{2}{\sqrt{29}}x + \frac{3}{\sqrt{29}}y + \frac{4}{\sqrt{29}}z = \frac{12}{\sqrt{29}}$$

This equation is of the form $lx + my + nz = d$, where l, m, n are the direction cosines of normal to the plane and d is the distance of normal from the origin.

The coordinates of the foot of the perpendicular are given by

$$(ld, md, nd).$$

$$\text{Therefore, the coordinates of the foot of the perpendicular are } \left(\frac{2}{\sqrt{29}} \cdot \frac{12}{\sqrt{29}}, \frac{3}{\sqrt{29}} \cdot \frac{12}{\sqrt{29}}, \frac{4}{\sqrt{29}} \cdot \frac{12}{\sqrt{29}} \right) \text{ i.e., } \left(\frac{24}{29}, \frac{36}{29}, \frac{48}{29} \right).$$

(b) Let the coordinates of the foot of perpendicular P from the origin to the plane be (x_1, y_1, z_1) .

$$3y + 4z - 6 = 0$$

$$\Rightarrow 0x + 3y + 4z = 6 \dots (1)$$

The direction ratios of the normal are 0, 3, and 4.

$$\therefore \sqrt{0^2 + 3^2 + 4^2} = 5$$

Dividing both sides of equation (1) by 5, we obtain

$$0x + \frac{3}{5}y + \frac{4}{5}z = \frac{6}{5}$$

This equation is of the form $lx + my + nz = d$, where l, m, n are the direction cosines of normal to the plane and d is the distance of normal from the origin.

The coordinates of the foot of the perpendicular are given by

$$(ld, md, nd).$$

Therefore, the coordinates of the foot of the perpendicular are

$$\left(0, \frac{3}{5} \cdot \frac{6}{5}, \frac{4}{5} \cdot \frac{6}{5} \right) \text{ i.e., } \left(0, \frac{18}{25}, \frac{24}{25} \right).$$

(c) Let the coordinates of the foot of perpendicular P from the origin to the plane be (x_1, y_1, z_1) .

$$x + y + z = 1 \dots (1)$$

The direction ratios of the normal are 1, 1, and 1.

$$\therefore \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

Dividing both sides of equation (1) by $\sqrt{3}$, we obtain

$$\frac{1}{\sqrt{3}}x + \frac{1}{\sqrt{3}}y + \frac{1}{\sqrt{3}}z = \frac{1}{\sqrt{3}}$$

This equation is of the form $lx + my + nz = d$, where l, m, n are the direction cosines of normal to the plane and d is the distance of normal from the origin.

The coordinates of the foot of the perpendicular are given by

$$(ld, md, nd).$$

Therefore, the coordinates of the foot of the perpendicular are

$$\left(\frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} \right) \text{ i.e., } \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right).$$

(d) Let the coordinates of the foot of perpendicular P from the origin to the plane be (x_1, y_1, z_1) .

$$5y + 8 = 0$$

$$\Rightarrow 0x - 5y + 0z = 8 \dots (1)$$

The direction ratios of the normal are 0, -5, and 0.

$$\therefore \sqrt{0^2 + (-5)^2 + 0^2} = 5$$

Dividing both sides of equation (1) by 5, we obtain

$$-y = \frac{8}{5}$$

This equation is of the form $lx + my + nz = d$, where l, m, n are the direction cosines of normal to the plane and d is the distance of normal from the origin.

The coordinates of the foot of the perpendicular are given by

$$(ld, md, nd).$$

Therefore, the coordinates of the foot of the perpendicular are

$$\left(0, -1\left(\frac{8}{5}\right), 0 \right) \text{ i.e., } \left(0, -\frac{8}{5}, 0 \right).$$

Question 5:

Find the vector and Cartesian equation of the planes

(a) that passes through the point $(1, 0, -2)$ and the normal to the plane is $\hat{i} + \hat{j} - \hat{k}$.

(b) that passes through the point $(1, 4, 6)$ and the normal vector to the plane is $\hat{i} - 2\hat{j} + \hat{k}$.

ANS :

(a) The position vector of point $(1, 0, -2)$ is $\vec{a} = \hat{i} - 2\hat{k}$

The normal vector \vec{N} perpendicular to the plane is $\vec{N} = \hat{i} + \hat{j} - \hat{k}$

The vector equation of the plane is given by, $(\vec{r} - \vec{a}) \cdot \vec{N} = 0$

$$\Rightarrow [\vec{r} - (\hat{i} - 2\hat{k})] \cdot (\hat{i} + \hat{j} - \hat{k}) = 0 \quad \dots(1)$$

\vec{r} is the position vector of any point P (x, y, z) in the plane.

$$\therefore \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

Therefore, equation (1) becomes

$$\begin{aligned} & [(x\hat{i} + y\hat{j} + z\hat{k}) - (\hat{i} - 2\hat{k})] \cdot (\hat{i} + \hat{j} - \hat{k}) = 0 \\ \Rightarrow & [(x-1)\hat{i} + y\hat{j} + (z+2)\hat{k}] \cdot (\hat{i} + \hat{j} - \hat{k}) = 0 \\ \Rightarrow & (x-1) + y - (z+2) = 0 \\ \Rightarrow & x + y - z - 3 = 0 \\ \Rightarrow & x + y - z = 3 \end{aligned}$$

This is the Cartesian equation of the required plane.

(b) The position vector of the point $(1, 4, 6)$ is $\vec{a} = \hat{i} + 4\hat{j} + 6\hat{k}$

The normal vector \vec{N} perpendicular to the plane is $\vec{N} = \hat{i} - 2\hat{j} + \hat{k}$

The vector equation of the plane is given by, $(\vec{r} - \vec{a}) \cdot \vec{N} = 0$

$$\Rightarrow [\vec{r} - (\hat{i} + 4\hat{j} + 6\hat{k})] \cdot (\hat{i} - 2\hat{j} + \hat{k}) = 0 \quad \dots(1)$$

\vec{r} is the position vector of any point P (x, y, z) in the plane.

$$\therefore \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

Therefore, equation (1) becomes

$$\begin{aligned} & [(x\hat{i} + y\hat{j} + z\hat{k}) - (\hat{i} + 4\hat{j} + 6\hat{k})] \cdot (\hat{i} - 2\hat{j} + \hat{k}) = 0 \\ \Rightarrow & [(x-1)\hat{i} + (y-4)\hat{j} + (z-6)\hat{k}] \cdot (\hat{i} - 2\hat{j} + \hat{k}) = 0 \\ \Rightarrow & (x-1) - 2(y-4) + (z-6) = 0 \\ \Rightarrow & x - 2y + z + 1 = 0 \end{aligned}$$

This is the Cartesian equation of the required plane.

Question 6:

Find the equations of the planes that passes through three points.

(a) $(1, 1, -1), (6, 4, -5), (-4, -2, 3)$

(b) $(1, 1, 0), (1, 2, 1), (-2, 2, -1)$

ANS :

(a) The given points are A $(1, 1, -1)$, B $(6, 4, -5)$, and C $(-4, -2, 3)$.

$$\begin{vmatrix} 1 & 1 & -1 \\ 6 & 4 & -5 \\ -4 & -2 & 3 \end{vmatrix} = (12 - 10) - (18 - 20) - (-12 + 16) \\ = 2 + 2 - 4 \\ = 0$$

Since A, B, C are collinear points, there will be infinite number of planes passing through the given points.

(b) The given points are A $(1, 1, 0)$, B $(1, 2, 1)$, and C $(-2, 2, -1)$.

$$\begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ -2 & 2 & -1 \end{vmatrix} = (-2 - 2) - (2 + 2) = -8 \neq 0$$

Therefore, a plane will pass through the points A, B, and C.

It is known that the equation of the plane through the points, (x_1, y_1, z_1) , (x_2, y_2, z_2) , and (x_3, y_3, z_3) , is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0 \\ \Rightarrow \begin{vmatrix} x - 1 & y - 1 & z \\ 0 & 1 & 1 \\ -3 & 1 & -1 \end{vmatrix} = 0$$

$$\Rightarrow (-2)(x - 1) - 3(y - 1) + 3z = 0$$

$$\Rightarrow -2x - 3y + 3z + 2 + 3 = 0$$

$$\Rightarrow -2x - 3y + 3z = -5$$

$$\Rightarrow 2x + 3y - 3z = 5$$

This is the Cartesian equation of the required plane.

Question 7:

Find the intercepts cut off by the plane $2x + y - z = 5$

ANS :

$$2x + y - z = 5 \quad \dots(1)$$

Dividing both sides of equation (1) by 5, we obtain

$$\begin{aligned} \frac{2}{5}x + \frac{y}{5} - \frac{z}{5} &= 1 \\ \Rightarrow \frac{x}{\frac{5}{2}} + \frac{y}{5} + \frac{z}{-5} &= 1 \quad \dots(2) \end{aligned}$$

It is known that the equation of a plane in intercept form is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$, where a, b, c are the intercepts cut off by the plane at x, y , and z axes respectively.

Therefore, for the given equation,

$$a = \frac{5}{2}, b = 5, \text{ and } c = -5$$

Thus, the intercepts cut off by the plane are $\frac{5}{2}, 5, \text{ and } -5$.

Question 8:

Find the equation of the plane with intercept 3 on the y -axis and parallel to ZOX plane.

ANS :

The equation of the plane ZOX is

$$y = 0$$

Any plane parallel to it is of the form, $y = a$

Since the y -intercept of the plane is 3,

$$\therefore a = 3$$

Thus, the equation of the required plane is $y = 3$

Question 9:

Find the equation of the plane through the intersection of the planes $3x - y + 2z - 4 = 0$ and $x + y + z - 2 = 0$ and the point $(2, 2, 1)$

ANS :

The equation of any plane through the intersection of the planes,

$3x - y + 2z - 4 = 0$ and $x + y + z - 2 = 0$, is

$$(3x - y + 2z - 4) + \alpha(x + y + z - 2) = 0, \text{ where } \alpha \in \mathbf{R} \quad \dots(1)$$

The plane passes through the point $(2, 2, 1)$. Therefore, this point will satisfy equation (1).

$$\therefore (3 \times 2 - 2 + 2 \times 1 - 4) + \alpha(2 + 2 + 1 - 2) = 0$$

$$\Rightarrow 2 + 3\alpha = 0$$

$$\Rightarrow \alpha = -\frac{2}{3}$$

Substituting $\alpha = -\frac{2}{3}$ in equation (1), we obtain

$$(3x - y + 2z - 4) - \frac{2}{3}(x + y + z - 2) = 0$$

$$\Rightarrow 3(3x - y + 2z - 4) - 2(x + y + z - 2) = 0$$

$$\Rightarrow (9x - 3y + 6z - 12) - 2(x + y + z - 2) = 0$$

$$\Rightarrow 7x - 5y + 4z - 8 = 0$$

This is the required equation of the plane.

Question 10:

Find the vector equation of the plane passing through the intersection of the planes $\vec{r} \cdot (2\hat{i} + 2\hat{j} - 3\hat{k}) = 7$, $\vec{r} \cdot (2\hat{i} + 5\hat{j} + 3\hat{k}) = 9$ and through the point $(2, 1, 3)$

ANS :

The equations of the planes are $\vec{r} \cdot (2\hat{i} + 2\hat{j} - 3\hat{k}) = 7$ and $\vec{r} \cdot (2\hat{i} + 5\hat{j} + 3\hat{k}) = 9$

$$\Rightarrow \vec{r} \cdot (2\hat{i} + 2\hat{j} - 3\hat{k}) - 7 = 0 \quad \dots(1)$$

$$\vec{r} \cdot (2\hat{i} + 5\hat{j} + 3\hat{k}) - 9 = 0 \quad \dots(2)$$

The equation of any plane through the intersection of the planes given in equations (1) and (2) is given by,

$$[\vec{r} \cdot (2\hat{i} + 2\hat{j} - 3\hat{k}) - 7] + \lambda[\vec{r} \cdot (2\hat{i} + 5\hat{j} + 3\hat{k}) - 9] = 0, \text{ where } \lambda \in \mathbf{R}$$

$$\vec{r} \cdot [(2\hat{i} + 2\hat{j} - 3\hat{k}) + \lambda(2\hat{i} + 5\hat{j} + 3\hat{k})] = 9\lambda + 7$$

$$\vec{r} \cdot [(2 + 2\lambda)\hat{i} + (2 + 5\lambda)\hat{j} + (3\lambda - 3)\hat{k}] = 9\lambda + 7 \quad \dots(3)$$

The plane passes through the point $(2, 1, 3)$. Therefore, its position vector is given by,

$$\vec{r} = 2\hat{i} + 1\hat{j} + 3\hat{k}$$

Substituting in equation (3), we obtain

$$(2\hat{i} + \hat{j} + 3\hat{k}) \cdot [(2 + 2\lambda)\hat{i} + (2 + 5\lambda)\hat{j} + (3\lambda - 3)\hat{k}] = 9\lambda + 7$$

$$\Rightarrow 2(2 + 2\lambda) + 1(2 + 5\lambda) + 3(3\lambda - 3) = 9\lambda + 7$$

$$\Rightarrow 4 + 4\lambda + 2 + 5\lambda + 9\lambda - 9 = 9\lambda + 7$$

$$\Rightarrow 18\lambda - 3 = 9\lambda + 7$$

$$\Rightarrow 9\lambda = 10$$

$$\Rightarrow \lambda = \frac{10}{9}$$

Substituting $\lambda = \frac{10}{9}$ in equation (3), we obtain

$$\vec{r} \cdot \left(\frac{38}{9}\hat{i} + \frac{68}{9}\hat{j} + \frac{3}{9}\hat{k} \right) = 17$$

$$\Rightarrow \vec{r} \cdot (38\hat{i} + 68\hat{j} + 3\hat{k}) = 153$$

This is the vector equation of the required plane.

Question 11:

Find the equation of the plane through the line of intersection of the planes $x + y + z = 1$ and $2x + 3y + 4z = 5$ which is perpendicular to the plane $x - y + z = 0$

ANS :

The equation of the plane through the intersection of the planes, $x + y + z = 1$ and $2x + 3y + 4z = 5$, is

$$(x + y + z - 1) + \lambda(2x + 3y + 4z - 5) = 0$$

$$\Rightarrow (2\lambda + 1)x + (3\lambda + 1)y + (4\lambda + 1)z - (5\lambda + 1) = 0 \quad \dots(1)$$

The direction ratios, a_1, b_1, c_1 , of this plane are $(2\lambda + 1)$, $(3\lambda + 1)$, and $(4\lambda + 1)$.

The plane in equation (1) is perpendicular to $x - y + z = 0$

Its direction ratios, a_2, b_2, c_2 , are 1, -1, and 1.

Since the planes are perpendicular,

$$a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$$

$$\Rightarrow (2\lambda + 1) - (3\lambda + 1) + (4\lambda + 1) = 0$$

$$\Rightarrow 3\lambda + 1 = 0$$

$$\Rightarrow \lambda = -\frac{1}{3}$$

Substituting $\lambda = -\frac{1}{3}$ in equation (1), we obtain

$$\begin{aligned}\frac{1}{3}x - \frac{1}{3}z + \frac{2}{3} &= 0 \\ \Rightarrow x - z + 2 &= 0\end{aligned}$$

This is the required equation of the plane.

Question 12:

Find the angle between the planes whose vector equations are

$$\vec{r} \cdot (2\hat{i} + 2\hat{j} - 3\hat{k}) = 5 \text{ and } \vec{r} \cdot (3\hat{i} - 3\hat{j} + 5\hat{k}) = 3.$$

ANS :

The equations of the given planes are $\vec{r} \cdot (2\hat{i} + 2\hat{j} - 3\hat{k}) = 5$ and $\vec{r} \cdot (3\hat{i} - 3\hat{j} + 5\hat{k}) = 3$

It is known that if \vec{n}_1 and \vec{n}_2 are normal to the planes, $\vec{r} \cdot \vec{n}_1 = d_1$ and $\vec{r} \cdot \vec{n}_2 = d_2$, then the angle between them, Q , is given by,

$$\cos Q = \frac{|\vec{n}_1 \cdot \vec{n}_2|}{|\vec{n}_1| |\vec{n}_2|} \quad \dots(1)$$

Here, $\vec{n}_1 = 2\hat{i} + 2\hat{j} - 3\hat{k}$ and $\vec{n}_2 = 3\hat{i} - 3\hat{j} + 5\hat{k}$

$$\therefore \vec{n}_1 \cdot \vec{n}_2 = (2\hat{i} + 2\hat{j} - 3\hat{k}) \cdot (3\hat{i} - 3\hat{j} + 5\hat{k}) = 2 \cdot 3 + 2 \cdot (-3) + (-3) \cdot 5 = -15$$

$$|\vec{n}_1| = \sqrt{(2)^2 + (2)^2 + (-3)^2} = \sqrt{17}$$

$$|\vec{n}_2| = \sqrt{(3)^2 + (-3)^2 + (5)^2} = \sqrt{43}$$

Substituting the value of $\vec{n}_1 \cdot \vec{n}_2$, $|\vec{n}_1|$ and $|\vec{n}_2|$ in equation (1), we obtain

$$\cos Q = \frac{|-15|}{\sqrt{17} \cdot \sqrt{43}}$$

$$\Rightarrow \cos Q = \frac{15}{\sqrt{731}}$$

$$\Rightarrow \cos Q^{-1} = \left(\frac{15}{\sqrt{731}} \right)$$

Question 13:

In the following cases, determine whether the given planes are parallel or perpendicular, and in case they are neither, find the angles between them.

- (a) $7x + 5y + 6z + 30 = 0$ and $3x - y - 10z + 4 = 0$
- (b) $2x + y + 3z - 2 = 0$ and $x - 2y + 5 = 0$
- (c) $2x - 2y + 4z + 5 = 0$ and $3x - 3y + 6z - 1 = 0$
- (d) $2x - y + 3z - 1 = 0$ and $2x - y + 3z + 3 = 0$
- (e) $4x + 8y + z - 8 = 0$ and $y + z - 4 = 0$

ANS :

The direction ratios of normal to the plane, $L_1 : a_1x + b_1y + c_1z = 0$, are a_1, b_1, c_1 and $L_2 : a_2x + b_2y + c_2z = 0$ are a_2, b_2, c_2 .

$$L_1 \parallel L_2, \text{ if } \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

$$L_1 \perp L_2, \text{ if } a_1a_2 + b_1b_2 + c_1c_2 = 0$$

The angle between L_1 and L_2 is given by,

$$Q = \cos^{-1} \left| \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2} \cdot \sqrt{a_2^2 + b_2^2 + c_2^2}} \right|$$

(a) The equations of the planes are $7x + 5y + 6z + 30 = 0$ and

$$3x - y - 10z + 4 = 0$$

$$\text{Here, } a_1 = 7, b_1 = 5, c_1 = 6$$

$$a_2 = 3, b_2 = -1, c_2 = -10$$

$$a_1a_2 + b_1b_2 + c_1c_2 = 7 \times 3 + 5 \times (-1) + 6 \times (-10) = -44 \neq 0$$

Therefore, the given planes are not perpendicular.

$$\frac{a_1}{a_2} = \frac{7}{3}, \frac{b_1}{b_2} = \frac{5}{-1} = -5, \frac{c_1}{c_2} = \frac{6}{-10} = \frac{-3}{5}$$

$$\text{It can be seen that, } \frac{a_1}{a_2} \neq \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$$

Therefore, the given planes are not parallel.

The angle between them is given by,

$$\begin{aligned} Q &= \cos^{-1} \left| \frac{7 \times 3 + 5 \times (-1) + 6 \times (-10)}{\sqrt{(7)^2 + (5)^2 + (6)^2} \times \sqrt{(3)^2 + (-1)^2 + (-10)^2}} \right| \\ &= \cos^{-1} \left| \frac{21 - 5 - 60}{\sqrt{110} \times \sqrt{110}} \right| \\ &= \cos^{-1} \frac{44}{110} \\ &= \cos^{-1} \frac{2}{5} \end{aligned}$$

(b) The equations of the planes are $2x + y + 3z - 2 = 0$ and $x - 2y + 5 = 0$

Here, $a_1 = 2, b_1 = 1, c_1 = 3$ and $a_2 = 1, b_2 = -2, c_2 = 0$

$$\therefore a_1 a_2 + b_1 b_2 + c_1 c_2 = 2 \times 1 + 1 \times (-2) + 3 \times 0 = 0$$

Thus, the given planes are perpendicular to each other.

(c) The equations of the given planes are $2x - 2y + 4z + 5 = 0$ and $3x - 3y + 6z - 1 = 0$

Here, $a_1 = 2, b_1 = -2, c_1 = 4$ and $a_2 = 3, b_2 = -3, c_2 = 6$ $a_1 a_2 + b_1 b_2 + c_1 c_2 = 2 \times 3 + (-2)(-3) + 4 \times 6 = 6 + 6 + 24 = 36 \neq 0$

Thus, the given planes are not perpendicular to each other.

$$\frac{a_1}{a_2} = \frac{2}{3}, \frac{b_1}{b_2} = \frac{-2}{-3} = \frac{2}{3} \text{ and } \frac{c_1}{c_2} = \frac{4}{6} = \frac{2}{3}$$

$$\therefore \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

Thus, the given planes are parallel to each other.

(d) The equations of the planes are $2x - y + 3z - 1 = 0$ and $2x - y + 3z + 3 = 0$

Here, $a_1 = 2, b_1 = -1, c_1 = 3$ and $a_2 = 2, b_2 = -1, c_2 = 3$

$$\frac{a_1}{a_2} = \frac{2}{2} = 1, \frac{b_1}{b_2} = \frac{-1}{-1} = 1 \text{ and } \frac{c_1}{c_2} = \frac{3}{3} = 1$$

$$\therefore \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

Thus, the given lines are parallel to each other.

(e) The equations of the given planes are $4x + 8y + z - 8 = 0$ and $y + z - 4 = 0$

Here, $a_1 = 4, b_1 = 8, c_1 = 1$ and $a_2 = 0, b_2 = 1, c_2 = 1$

$$a_1 a_2 + b_1 b_2 + c_1 c_2 = 4 \times 0 + 8 \times 1 + 1 = 9 \neq 0$$

Therefore, the given lines are not perpendicular to each other.

$$\frac{a_1}{a_2} = \frac{4}{0}, \frac{b_1}{b_2} = \frac{8}{1} = 8, \frac{c_1}{c_2} = \frac{1}{1} = 1$$

$$\therefore \frac{a_1}{a_2} \neq \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$$

Therefore, the given lines are not parallel to each other.

The angle between the planes is given by,

$$Q = \cos^{-1} \left| \frac{4 \times 0 + 8 \times 1 + 1 \times 1}{\sqrt{4^2 + 8^2 + 1^2} \times \sqrt{0^2 + 1^2 + 1^2}} \right| = \cos^{-1} \left| \frac{9}{9 \times \sqrt{2}} \right| = \cos^{-1} \left(\frac{1}{\sqrt{2}} \right) = 45^\circ$$

Question 14:

In the following cases, find the distance of each of the given points from the corresponding given plane.

Point Plane

(a) $(0, 0, 0)$ $3x - 4y + 12z = 3$

(b) $(3, -2, 1)$ $2x - y + 2z + 3 = 0$

(c) $(2, 3, -5)$ $x + 2y - 2z = 9$

(d) $(-6, 0, 0)$ $2x - 3y + 6z - 2 = 0$

ANS :

It is known that the distance between a point, $p(x_1, y_1, z_1)$, and a plane, $Ax + By + Cz = D$, is given by,

$$d = \left| \frac{Ax_1 + By_1 + Cz_1 - D}{\sqrt{A^2 + B^2 + C^2}} \right| \quad \dots(1)$$

(a) The given point is $(0, 0, 0)$ and the plane is $3x - 4y + 12z = 3$

$$\therefore d = \left| \frac{3 \times 0 - 4 \times 0 + 12 \times 0 - 3}{\sqrt{(3)^2 + (-4)^2 + (12)^2}} \right| = \frac{3}{\sqrt{169}} = \frac{3}{13}$$

(b) The given point is $(3, -2, 1)$ and the plane is $2x - y + 2z + 3 = 0$

$$\therefore d = \left| \frac{2 \times 3 - (-2) + 2 \times 1 + 3}{\sqrt{(2)^2 + (-1)^2 + (2)^2}} \right| = \left| \frac{13}{3} \right| = \frac{13}{3}$$

(c) The given point is $(2, 3, -5)$ and the plane is $x + 2y - 2z = 9$

$$\therefore d = \left| \frac{2 + 2 \times 3 - 2(-5) - 9}{\sqrt{(1)^2 + (2)^2 + (-2)^2}} \right| = \frac{9}{3} = 3$$

(d) The given point is $(-6, 0, 0)$ and the plane is $2x - 3y + 6z - 2 = 0$

$$d = \left| \frac{2(-6) - 3 \times 0 + 6 \times 0 - 2}{\sqrt{(2)^2 + (-3)^2 + (6)^2}} \right| = \left| \frac{-14}{\sqrt{49}} \right| = \frac{14}{7} = 2$$