ENRICHMENT ASSIGNMENT
CLASS XII

RELATIONS AND FUNCTIONS

Q1. Given a function defined by $y = f(x) = \sqrt{4-x^2}$; $0 \le x \le 2, 0 \le y \le 2$. Find the inverse of f.

Q2. If $f: R \to R$ be given by $f(x) = \sin^2 x + \sin^2(x+\pi/3) + \cos x \cos(x+\pi/3)\ \forall\ x \in R$, and $g: R \to R$ be such that $g(5/4) = 1$, then prove that $gof: R \to R$ is a constant function.

Q3. If the function $f: [1,\infty) \to [1,\infty)$ defined by $f(x) = 2^{x(x-1)}$ is invertible, find $f^{-1}(x)$.

Ans: $f^{-1}(x) = \dfrac{1+\sqrt{1+4\log_2 x}}{2}$

Q4. Find the value of parameter α for which the function $f(x) = 1 + \alpha x$, $\alpha \ne 0$ is the inverse of itself.

Ans: $\alpha = -1$

Q5. Let C and R denote the set of all complex numbers ans all real numbers respectively. Show that $f: C \to R$ given by $f(z) = |z|\ \forall\ z \in C$ is neither one-one nor onto.

Q6. Show that $f: R \to R$ given by $f(x) = \dfrac{x}{x^2+1}, \forall\, x \in R$ is neither one-one nor onto.

Q7. Let $f, g: R \to R$ be two functiones defined as $f(x) = |x| + x$ and $g(x) = |x| - x, \forall\, x \in R$. Find fog and gof.

Ans: $(gof)(x) = 0$, $(fog)(x) = \begin{cases} 0 & , x > 0 \\ -4x & , x < 0 \end{cases}$

Q8. On the set $M = A(x) = \left\{ \begin{bmatrix} x & x \\ x & x \end{bmatrix} : x \in R \right\}$ of 2×2 matrices, find the identity element for the multiplication of matrices as a binary operation. Also, find the inverse of an element of M.

Ans: $A\left(\dfrac{1}{2}\right) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ is identity element, $A(y) = A\left(\dfrac{1}{4x}\right) = \begin{bmatrix} \frac{1}{4x} & \frac{1}{4x} \\ \frac{1}{4x} & \frac{1}{4x} \end{bmatrix}$ is the inverse.

ENRICHMENT ASSIGNMENT

INVERSE TRIGONOMETRY

Q1. Prove that $2\tan^{-1}\left(\sqrt{\dfrac{a-b}{a+b}}\tan\dfrac{\theta}{2}\right)=\cos^{-1}\left(\dfrac{b+a\cos\theta}{a+b\cos\theta}\right)$.

Q2. If $\cos^{-1}\dfrac{x}{a}+\cos^{-1}\dfrac{y}{b}=\theta$, prove that $\dfrac{x^2}{a^2}-\dfrac{2xy}{ab}\cos\theta+\dfrac{y^2}{b^2}=\sin^2\theta$.

Q3. If $\tan^{-1}\left(\dfrac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right)=\theta$, prove that $x^2=\sin 2\theta$.

Q4. Prove that: $\tan^{-1}\dfrac{1}{3}+\tan^{-1}\dfrac{1}{7}+.....+\tan^{-1}\dfrac{1}{n^2+n+1}=\tan^{-1}\dfrac{n}{n+2}$.

Q5. Solve the following equation: $\tan^{-1}\dfrac{1}{1+2x}+\tan^{-1}\dfrac{1}{4x+1}=\tan^{-1}\dfrac{2}{x^2}$. Ans: $3\,,\,-\dfrac{2}{3}$

Q6. Prove that : $\sin^{-1}\dfrac{4}{5}+\sin^{-1}\dfrac{5}{13}+\sin^{-1}\dfrac{16}{65}=\dfrac{\pi}{2}$.

Q7. Solve the following equation: $\sin^{-1}6x+\sin^{-1}\left(6\sqrt{3}x\right)=\dfrac{\pi}{2}$ Ans: $x=\dfrac{1}{12}$

Q8. If $\sin^{-1}x+\sin^{-1}y+\sin^{-1}z=\pi$, prove that $x\sqrt{1-x^2}+y\sqrt{1-y^2}+z\sqrt{1-z^2}=2xyz$.

Q9. Find all positive integral solutions of: $\tan^{-1}x+\cos^{-1}\left(\dfrac{y}{\sqrt{1+y^2}}\right)=\sin^{-1}\left(\dfrac{3}{\sqrt{10}}\right)$ Ans: $(1,2),(2,7)$

Q10. Solve for x : $3\sin^{-1}\left(\dfrac{2x}{1+x^2}\right)-4\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)+2\tan^{-1}\left(\dfrac{2x}{1-x^2}\right)=\dfrac{\pi}{3}$ Ans: $\dfrac{1}{\sqrt{3}}$

Q11. If $x,y,z\in[-1,1]$ such that $\sin^{-1}x+\sin^{-1}y+\sin^{-1}z=\dfrac{3\pi}{2}$, find the value of $x^{2009}+y^{2010}+z^{2011}-\dfrac{9}{x^{2009}+y^{2010}+z^{2011}}$. Ans: 0

Q12. Prove that : (a) $\sec^2\left(\tan^{-1}2\right)+\operatorname{cosec}^2\left(\cot^{-1}3\right)=15$ (b) $\cos\left[\tan^{-1}\left\{\sin\left(\cot^{-1}x\right)\right\}\right]=\sqrt{\dfrac{x^2+1}{x^2+2}}$

Q13. Prove that $\tan\left\{\dfrac{\pi}{4}+\dfrac{1}{2}\cos^{-1}\dfrac{a}{b}\right\}+\tan\left\{\dfrac{\pi}{4}-\dfrac{1}{2}\cos^{-1}\dfrac{a}{b}\right\}=\dfrac{2b}{a}$.

Q14. If $\sin^{-1}\frac{2a}{1+a^2} + \sin^{-1}\frac{2b}{1+b^2} = 2\tan^{-1}x$, prove that $x = \frac{a+b}{1-ab}$.

Q15. Evaluate the following: (a) $\sin^{-1}(\sin 10)$ (b) $\cos^{-1}(\cos 10)$ (c) $\tan^{-1}\{tan(-6)\}$

Ans: (a) $3\pi - 10$ (b) $4\pi - 10$ (c) $2\pi - 6$

Q16. Solve for x and y: $\sin^{-1}x + \sin^{-1}y = \frac{2\pi}{3}$, $\cos^{-1}x - \cos^{-1}y = \frac{\pi}{3}$ Ans: $x = \frac{1}{2}$, $y = 1$

Q17. Prove the following: $\cot^{-1}\left(\frac{ab+1}{a-b}\right) + \cot^{-1}\left(\frac{bc+1}{b-c}\right) + \cot^{-1}\left(\frac{ca+1}{c-a}\right) = 0$.

Q18. Show that: $\sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right) + \cos^{-1}\left(\frac{x+1}{\sqrt{x^2+2x+2}}\right) = \tan^{-1}(x^2+x+1)$.

Q19. Find the value of $\sum_{x=0}^{\infty}\tan^{-1}\left(\frac{1}{1+x+x^2}\right)$. Ans: $\frac{\pi}{2}$

Q20. Find the greatest and least values of $(\sin^{-1}x)^2 + (\cos^{-1}x)^2$. Ans: Least$=\frac{\pi^2}{8}$;Greatest$=\frac{5\pi^2}{4}$

Q21. If $\tan^{-1}\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}} = \frac{\pi}{4} - \frac{\alpha}{2}$, show that $x = \sin^2\alpha$.

Q22. If $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$, prove that $x^2 + y^2 + z^2 + 2xyz = 1$

ENRICHMENT ASSIGNMENT

MATRICES AND DETERMINANTS

Q1. If $A = \begin{bmatrix} \cos\theta + \sin\theta & \sqrt{2}\sin\theta \\ -\sqrt{2}\sin\theta & \cos\theta - \sin\theta \end{bmatrix}$, then prove by principle of mathematical induction that $A = \begin{bmatrix} \cos n\theta + \sin\theta & \sqrt{2}\sin n\theta \\ -\sqrt{2}\sin n\theta & \cos n\theta - \sin n\theta \end{bmatrix}$, where $n \in N$.

Q2. If $A = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$, show that $A^T A^{-1} = \begin{bmatrix} \cos 2x & -\sin 2x \\ \sin 2x & \cos 2x \end{bmatrix}$.

Q3. Find a matrix A such that $\begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix} A = \begin{bmatrix} -1 & -8 & -10 \\ 1 & -2 & -5 \\ 9 & 22 & 15 \end{bmatrix}$. Ans: $A = \begin{bmatrix} 1 & -2 & -5 \\ 3 & 4 & 0 \end{bmatrix}$

Q4. A triangle has its three sides equal to a, b and c. If the coordinates of its vertices are $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$, prove that $\begin{vmatrix} x_1 & y_1 & 2 \\ x_2 & y_2 & 2 \\ x_3 & y_3 & 2 \end{vmatrix}^2 = (a+b+c)(b+c-a)(c+a-b)(a+b-c)$.

Q5. Show that $\begin{vmatrix} a & b-c & c+b \\ a+c & b & c-a \\ a-b & b+a & c \end{vmatrix} = (a+b+c)(a^2+b^2+c^2)$.

Q6. (a) If $m \in N$ and $m \geq 2$, prove that $\begin{vmatrix} 1 & 1 & 1 \\ {}^{m}C_1 & {}^{m+1}C_1 & {}^{m+2}C_1 \\ {}^{m}C_2 & {}^{m+1}C_2 & {}^{m+2}C_2 \end{vmatrix} = 1$.

(b) Prove that: $\begin{vmatrix} \sqrt{13}+\sqrt{3} & 2\sqrt{5} & \sqrt{5} \\ \sqrt{15}+\sqrt{26} & 5 & \sqrt{10} \\ 3+\sqrt{65} & \sqrt{15} & 5 \end{vmatrix} = 5\sqrt{3}\left(\sqrt{6}-5\right)$

Q7. If $\Delta_r = \begin{vmatrix} r & x & \dfrac{n(n+1)}{2} \\ 2r-1 & y & n^2 \\ 3r-2 & z & \dfrac{n(3n-1)}{2} \end{vmatrix}$, show that $\sum_{r=1}^{n} \Delta_r = 0$.

Q8. Find the value of θ satisfying $\begin{vmatrix} 1 & 1 & \sin 3\theta \\ -4 & 3 & \cos 2\theta \\ 7 & -7 & -2 \end{vmatrix} = 0$. Ans: $\theta = n\pi$ or $n\pi + (-1)^n \dfrac{\pi}{6}$; $n \in N$

Q9. Prove that $\Delta = \begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ca-b^2 \end{vmatrix}$ is divisible by $a+b+c$ and find the quotient.

Ans: $\left(a^2+b^2+c^2-ab-bc-ca\right)^2$.

Q10. In a triangle ABC, if $\begin{vmatrix} 1 & 1 & 1 \\ 1+\sin A & 1+\sin B & 1+\sin C \\ \sin A+\sin^2 A & \sin B+\sin^2 B & \sin C+\sin^2 C \end{vmatrix} = 0$, prove that ΔABC is an isosceles triangle.

Q11. If $\begin{vmatrix} x+a & a^2 & a^3 \\ x+b & b^2 & b^3 \\ x+c & c^2 & c^3 \end{vmatrix} = 0$ and $a \neq b \neq c$, then show that $x = -\dfrac{abc}{ab+bc+ca}$.

Q12. Prove that: $\begin{vmatrix} {}^xC_1 & {}^xC_2 & {}^xC_3 \\ {}^yC_1 & {}^yC_2 & {}^yC_3 \\ {}^zC_1 & {}^zC_2 & {}^zC_3 \end{vmatrix} = \dfrac{xyz}{12}(x-y)(y-z)(z-x)$.

Q13. Solve using matrices: $x-y+z=3$, $2x+y-z=2$, $-x-2y+2z=1$.

Ans: $x=\dfrac{5}{3}$, $y=\dfrac{3k-4}{3}$, $z=k \quad \forall\ k \in R$

Q14. Solve using matrices: $3x-y+2z=3$, $2x+y+3z=5$, $x-2y-z=1$. Ans: No solution

Q15. Solve the following system of equations by using determinants:

$$x+y+z=1$$
$$ax+by+cz=k$$
$$a^2x+b^2y+c^2z=k^2$$

Ans: $x=\dfrac{(c-k)(k-b)}{(c-a)(a-b)}$, $y=\dfrac{(k-c)(a-k)}{(b-c)(a-b)}$, $z=\dfrac{(b-k)(k-a)}{(b-c)(c-a)}$

Q16. For what values of a and b, the system of equations $\begin{matrix} 2x+ay+6z=8 \\ x+2y+bz=5 \\ x+y+3z=4 \end{matrix}$ has:

(a) a unique solution — Ans: $a \neq 2$ or $b \neq 3$

(b) infinitely many solutions — Ans: $a=2$

(c) no solution — Ans: $b=3$

Q17. Solve the following system of homogeneous equations:

$x+y-z=0$, $x-2y+z=0$, $3x+6y-5z=0$

Ans: $x=\dfrac{k}{3}$, $y=\dfrac{2k}{3}$, $z=k$ for any k

<u>ENRICHMENT ASSIGNMENT</u>

CONTINUITY AND DIFFERENTIABILITY

Q1. Discuss the continuity of the function $f(x)=\begin{cases}\dfrac{x}{|x|+2x^2} & ,x\neq 0\\ 2 & ,x=0\end{cases}$ at $x=0$. Ans: discontinuous

Q2. Show that the function $f(x)=\begin{cases}\dfrac{e^{1/x}-1}{e^{1/x}+1} & ,x\neq 0\\ 0 & ,x=0\end{cases}$ is discontinuous at $x=0$.

Q3. If $f(x)=\begin{cases}\dfrac{\sin(a+1)x+\sin x}{x} & ,x<0\\ c & ,\text{if } x=0\\ \dfrac{\sqrt{x+bx^2}-\sqrt{x}}{bx^{3/2}} & ,\text{if } x>0\end{cases}$ is continous at $x=0$, find the value of a,b and c.

Ans: $a=-\dfrac{3}{2}, c=\dfrac{1}{2}$, and b may have any real value.

Q4. If $f(x)=\begin{cases}\dfrac{\log(1+ax)-\log(1-bx)}{x} & ,x\neq 0\\ k & ,x=0\end{cases}$ is continous at $x=0$, find the value of k. Ans: $a+b$

Q5. If $f(x)$ is continuous on its domain where $f(x)=\begin{cases}x^2+ax+b & ,0\le x<2\\ 4x-1 & ,2\le x\le 4\\ ax^2+17b & ,4<x\le 6\end{cases}$, find a and b.

Ans: $a=2, b=-1$

Q6. Let $f(x)=\begin{cases}\dfrac{1-\cos 4x}{x^2} & \text{if } x<0\\ a & \text{if } x=0\\ \dfrac{\sqrt{x}}{\sqrt{16+\sqrt{x}}-4} & \text{if } x>0\end{cases}$. Find a so that $f(x)$ is continuous at $x=0$. Ans: $a=8$

Q7. If $f(x)=\begin{cases}\dfrac{x-4}{|x-4|}+a & \text{if } x<4\\ a+b & \text{if } x=4\\ \dfrac{x-4}{|x-4|}+b & \text{if } x>4\end{cases}$, is continous at $x=4$, find a and b. Ans: $a=1$, $b=-1$

Q8. If $f(x)=\begin{cases}\dfrac{1-\sin^3 x}{3\cos^2 x} & if\ \ x<\dfrac{\pi}{2}\\ a & if\ \ x=\dfrac{\pi}{2}\\ \dfrac{b(1-\sin x)}{(\pi-2x)^2} & if\ \ x>\dfrac{\pi}{2}\end{cases}$ is continous at $x=\dfrac{\pi}{2}$, find a and b. Ans: $a=\dfrac{1}{2}$, $b=4$

Q9. Show that $f(x)=\begin{cases}x^2\sin\dfrac{1}{x} & , x\neq 0\\ 0 & , x=0\end{cases}$, is differentiable at $x=0$.

Q10. The function $f(x)=\begin{cases}x^2+3x+a & , x\leq 1\\ bx+2 & , x>1\end{cases}$ is given to be differentiable for every x. Find a and b. Ans: $a=3$, $b=5$

Q11. Show that $f(x)=\begin{cases}3x-2 & ,0<x\leq 1\\ 2x^2-x & ,1<x\leq 2\\ 5x-4 & ,x>2\end{cases}$ is continous at $x=2$, but not differentiable thereat.

<u>ENRICHMENT ASSIGNMENT</u>

DIFFERENTIATION

Q1. Differentiate the following:

(a) $\tan^{-1}\left(\dfrac{a\cos x-b\sin x}{b\cos x+a\sin x}\right)$ (b) $\tan^{-1}\left(\dfrac{ax-b}{bx+a}\right)$ (c)

Ans: $(a)\ -1$ $(b)\dfrac{1}{1+x^2}$

Q2. If $y=\tan^{-1}\left(\dfrac{4x}{1+5x^2}\right)+\tan^{-1}\left(\dfrac{2+3x}{3-2x}\right)$, prove that $\dfrac{dy}{dx}=\dfrac{5}{1+25x^2}$.

Q3. If $y=\sqrt{\sin x+\sqrt{\sin x+\sqrt{\sin x+\ldots\infty}}}$, prove that $\dfrac{dy}{dx}=\dfrac{\cos x}{2y-1}$.

Q4. If $y=(\cos x)^{(\cos x)^{(\cos x)\ldots\infty}}$, prove that $\dfrac{dy}{dx}=\dfrac{-y^2\tan x}{1-y\log\cos x}$.

Q5. If $y = x + \cfrac{1}{x + \cfrac{1}{x + \cfrac{1}{x + \ldots\infty}}}$, show that $\frac{dy}{dx} = \frac{y}{2y - x}$.

Q6. If $\cos^{-1}\left(\frac{x^2 - y^2}{x^2 + y^2}\right) = \tan^{-1} a$, prove that $\frac{dy}{dx} = \frac{y}{x}$.

Q7. Given that $\cos\frac{x}{2}.\cos\frac{x}{4}.\cos\frac{x}{8}.....= \frac{\sin x}{x}$, prove that $\frac{1}{2^2}\sec^2\frac{x}{2} + \frac{1}{2^4}\sec^2\frac{x}{4} + ... = \operatorname{cosec}^2 x - \frac{1}{x^2}$.

Q8. Diferentiate $\tan^{-1}\left(\frac{x}{1 + \sqrt{1 - x^2}}\right)$ w.r.t. $\sin\left(2\cot^{-1}\sqrt{\frac{1+x}{1-x}}\right)$.

Q9. If $y = \frac{ax^2}{(x-a)(x-b)(x-c)} + \frac{bx}{(x-b)(x-c)} + \frac{c}{x-c} + 1$, prove that $\frac{y'}{y} = \frac{1}{x}\left(\frac{a}{a-x} + \frac{b}{b-x} + \frac{c}{c-x}\right)$.

Q10. It is given that for the function f given by $f(x) = x^3 + bx^2 + ax$, $x \in [1,3]$ Rolle's Theorem holds with $c = 2 + \frac{1}{\sqrt{3}}$. Find the values of a and b. Ans: $a = 11$, $b = -6$

Q11. Verify Lagrange's Mean Value Theorem for $f(x) = (x-3)(x-6)(x-9)$ in the interval $[3,5]$.

ENRICHMENT ASSIGNMENT

APPLICATION OF DERIVATIVES

Q1. The curve $y = ax^3 + bx^2 + cx + 5$ touches the x-axis at $P(-2,0)$ and cuts the y-axis at the point Q where its gradient is 3. Find the equation of the curve. Ans: $y = -\frac{1}{2}x^3 - \frac{3}{4}x^2 + 3x + 5$

Q2. Find the intervals in which $f(x) = 2\log(x-2) - x^2 + 4x + 1$ is increasing or decreasing.

Ans: increasing on $(2,3)$ and decreasing on $(3,\infty)$

Q3. If a, b, c are real numbers, then find the intervals , in which $f(x)=\begin{vmatrix} x+a^2 & ab & ac \\ ab & x+b^2 & bc \\ ac & bc & x+c^2 \end{vmatrix}$ is increasing or decreasing.

Ans: increasing on $\left(-\infty, -\frac{2}{3}\left(a^2+b^2+c^2\right)\right) \cup (0, \infty)$, decreasing on $\left(-\frac{2}{3}\left(a^2+b^2+c^2\right), 0\right)$

Q4. Find the intervals in which $f(x) = \sin^4 x + \cos^4 x$, $0 \le x \le \frac{\pi}{2}$ is strictly increasing or decreasing.

Ans: increasing in $\left(\frac{\pi}{4}, \frac{\pi}{2}\right]$, decreasing in $\left[0, \frac{\pi}{4}\right)$

Q5. If $y=\frac{ax-b}{(x-1)(x-4)}$ has a turning point at $P(2,-1)$, find the values of a and b and show that y is maximum at P. $a = 1$, $b = 0$

Q6. Find the angle of intersection of the curves $y^2 = 4x$ and $x^2 = 4y$. Ans: $\frac{\pi}{2}$ or $\tan^{-1}\frac{3}{4}$

Q7. Prove that the line $\frac{x}{a} + \frac{y}{b} = 1$ is a tangent to curve $y = be^{-\frac{x}{a}}$, at a point where curve cuts y-axis.

Q8. Prove that $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2$ touches the straight line $\frac{x}{a}+\frac{y}{b}=2$ at (a, b) $\forall$ n.

Q9. Find the points of local maximum and local minimum of $f(x) = x\sqrt{1-x}$, $x \le 1$. Find also the local maximum and local mimimum value. Ans: Laocal max. at $x = \frac{2}{3}$, value is $\frac{2}{3\sqrt{3}}$

Q10. If $A, B > 0$ and $A+B=\frac{\pi}{3}$, then prove that the maximum value of $\tan A \tan B$ is $\frac{1}{3}$.