SAMPLE QUESTIONS FOR SECOND INTRA SCHOOL MATHEMATICS OLYMPIAD 2011

CLASS XII

QUESTIONS

1.

What is the sum of all values of x that satisfy the equation $\left(x^2-5x+5\right)^{x^2+4x-60} - 1$? Hint: There are three cases to consider.

2.

What is the length of the curve that is built by infinitely many semicircles with radius 1, 1/2, 1/4, etc. according to the picture: 

3.

What is the probability that out of three friends, exactly two have the same birth month? (assume that the 12 birth months have equal probabilities).

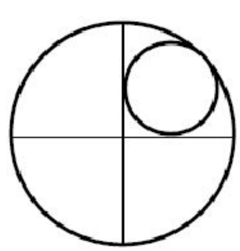
4.

The positive integers are arranged in increasing order in a triangle, as shown. Each row contains one more number than the previous row. The sum of the numbers in the row that contains the number 400 is

1
2 3
4 5 6
7 8 9 10
11 12 ...

5.

Two perpendicular lines, intersecting at the center of a circle of radius 1cm, divide the circle into four parts. A smaller circle is inscribed in one of those parts as shown. What is the radius of the smaller circle?



SOLUTIONS

1.

There are three cases to consider for $a^b = 1$, where $a = x^2 - 5x + 5$ and $b = x^2 - 4x - 60$.
Case 1: The base, *a*, equals 1. Thus

$x^2 - 5x + 5 = 1$

$x^2 - 5x + 4 = 0$

$(x-4)(x-1) = 0$

$x = 4$ or $x = 1$ will result in a value of 1 to either the exponent -28 or -55 so that $a^b = 1$.

Case 2: The exponent, *b*, equals 0 and $a \neq 0$ since this would give a value of 0^0 which is indeterminate. Thus

$x^2 + 4x - 60 = 0$

$(x+10)(x-6) = 0$

$x = -10$ or $x = 6$ will give 155^0 or 11^0 so that $a^b = 1$.

Case 3: The base, *a*, equals -1 to an even exponent. Thus

$x^2 - 5x + 5 = -1$

$x^2 - 5x + 6 = 0$

$(x-2)(x-3) = 0$

$x = 2$ or $x = 3$

Now we need to check whether these values of x will give an even exponent:

$x^2 + 4x - 60 = 0$

$(2)^2 + 4(2) - 60 = -48$

and

$x^2 + 4x - 60 = 0$

$(3)^2 + 4(3) - 60 = -39$

$x = 2$ will give $(-1)^{-48} = 1$, so $x = 2$ is a solution. When $x = 3$ we get $(-1)^{-39} = -1 \neq 1$.
Thus only $x = 2$ will give an even exponent.

Therefore the sum of the values that satisfy the equation $\left(x^2 - 5x + 5\right)^{x^2+4x-60} = 1$ is

$4 + 1 - 10 + 6 + 2 = 3$.

2.

The first semicircle has length of π, the second has length of $\frac{1}{2}\pi$, the third has length of $\frac{1}{4}\pi$, etc. The total length is

$$\pi + \frac{1}{2}\pi + \frac{1}{4}\pi + \ldots = \pi(1 + \frac{1}{2} + \frac{1}{4} + \ldots) = 2\pi,$$

where the last summation can be done using the formula for the sum of a geometric sequence, or simply by visualizing the pieces next to each other adding up to 2.

3.

I. For the first two out of three to have the same birth month, the first friend can have any month (probability 12/12) a second friend must have the same birth month (probability 1/12) and a third must have a different birth month from the other two (probability 11/12). To find the overall probability we multiply these probabilities together and get $1/12 \times 11/12 = 11/144$. Similarly to friends 1 & 2; 1 & 3, or 2 & 3 can have the same birth month. So we must multiply this result by 3 to get the final probability of 11/48.

II. Another way to solve the problem is to look at the complement of the event of exactly two friends having the same birth month. It means either all three having the same birth month (the probability of which is $12/12 \times 1/12 \times 1/12 = 1/144$), or all of them have different birth months (with probability of $12/12 \times 11/12 \times 10/12 = 110/144$). The probability of exactly two of them to have the same birth month is then $1 - 1/144 - 110/144 = 33/144 = 11/48$.

4.

In the given pattern, the rth row contains r integers.
Therefore, after n rows, the total number of integers appearing in the pattern is

$$1 + 2 + 3 + \cdots + (n-2) + (n-1) + n$$

This expression is always equal to $\frac{1}{2}n(n+1)$.
(If you have never seen this formula before, try to prove it!)
Putting this another way, the largest number in the nth row is $\frac{1}{2}n(n+1)$.
To determine which row the number 400 is in, we want to determine the smallest value of n for which $\frac{1}{2}n(n+1) \geq 400$ or $n(n+1) \geq 800$.
If $n = 27$, then $n(n+1) = 756$.
If $n = 28$, then $n(n+1) = 812$.
Therefore, 400 appears in the 28th row. Also, the largest integer in the 28th row is 406 and the largest integer in the 27th row is 378.
Thus, we want to determine the sum of the integers from 379 (the first integer in the 28th row) to 406, inclusive.
We can do this by calculating the sum of the integers from 1 to 406 and subtracting the sum of the integers from 1 to 378.
Since the sum of the integers from 1 to m equals $\frac{1}{2}m(m+1)$, then the sum of the integers from 379 to 406 is equal to $\frac{1}{2}(406)(407) - \frac{1}{2}(378)(379) = 10990$.

5.

Let x denote the radius of the smaller circle. Then the figure has lengths as shown. It follows that:

$$x^2 + x^2 = (1-x)^2$$
$$\Rightarrow x^2 + 2x - 1 = 0$$
$$\Rightarrow x = \pm\sqrt{2} - 1$$

Since $x > 0$, we obtain $x = \sqrt{2} - 1$.