SAMPLE QUESTIONS FOR SECOND INTRA SCHOOL MATHEMATICS OLYMPIAD 2011

CLASS VIII

QUESTIONS

1.

In the late 1700s, Gauss was asked to find the sum of the numbers from 1 to 100. Gauss quickly gave the answer 5 050. He did this by looking at patterns. Instead of finding the sum of the numbers 1 to 100, can you find the sum of the **digits** of the numbers from 1 to 100?

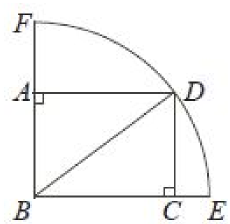
1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

2.

8^3 means $8 \times 8 \times 8$ and equals 512 when expressed as an integer. When 8^{2011} is expressed as an integer, what is its last digit?

3.

$ABCD$ is a rectangle inscribed in a quarter-circle as shown. A is on BF, B is the centre of the quarter-circle, C is on BE, and D is on arc FE. If $AD = 12$ cm and $CE = 1$ cm, determine the length of AF.

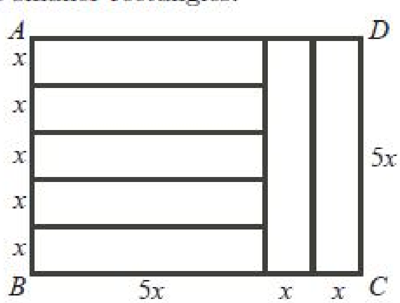


4.

Three positive numbers exist such that the following is true: the product of the first and second numbers equals the third number; the product of the second and third numbers is 180; and the second number is five times the third number. Determine the product of the three numbers.

5.

Seven identical rectangles are arranged as shown in the diagram to form a large rectangle $ABCD$. If the area of rectangle $ABCD$ is 560 cm^2, determine the dimensions of the smaller rectangles.



SOLUTIONS

1.

(1) Each of the ten columns has a units digit that occurs ten times.

$$\begin{aligned}\text{Sum of ALL units digits} &= 10(1)+10(2)+10(3)+\cdots+10(9)+10(0)\\ &= 10(1+2+3+\cdots+9+0)\\ &= 10(45)\\ &= 450\end{aligned}$$

(2) Each of the ten columns has a ten's digit from 0 to 9.

$$\begin{aligned}\text{Sum of ALL tens digits} &= 10(0+1+2+3+\cdots+8+9)\\ &= 10(45)\\ &= 450\end{aligned}$$

(3) The number 100 is the only number with a hundreds digit. We need to add 1 to our final sum.

(4) Now we add our results from (1), (2), and (3) to obtain the required sum.

$$\begin{aligned}\text{Sum of digits} &= \text{Units digit sum} + \text{Tens digit sum} + \text{Hundreds Digit}\\ &= 450+450+1\\ &= 901\end{aligned}$$

Therefore the sum of the digits of the numbers from 1 to 100 is 901.

2.

Let's start by examining the last digit of various powers of 8.

$$\begin{aligned}8^1 &= 8\\ 8^2 &= 64\\ 8^3 &= 512\\ 8^4 &= 4\ 096\\ 8^5 &= 32\ 768\\ 8^6 &= 262\ 144\\ 8^7 &= 2\ 097\ 152\\ 8^8 &= 16\ 777\ 216\end{aligned}$$

Notice that the last digit repeats every four powers of 8. The pattern continues. 8^9 ends with 8, 8^{10} ends with 4, 8^{11} ends with 2, 8^{12} ends with 6, and so on. Starting with the first power of 8, every four consecutive powers of 8 will have the last digit 8, 4, 2, and 6.

We need to determine the number of complete cycles by dividing 2011 by 4.

$$\frac{2011}{4} = 502\frac{3}{4}$$

There are 502 complete cycles and $\frac{3}{4}$ of another cycle. $502 \times 4 = 2008$ so 8^{2008} is the last power of 8 in the 502nd cycle and therefore ends with 6.

To go $\frac{3}{4}$ of the way into the next cycle tells us that the number 8^{2011} ends with the third number in the pattern, namely 2. In fact, we know that 8^{2009} ends with 8, 8^{2010} ends with 4, 8^{2011} ends with 2, and 8^{2012} ends with 6 because they would be the numbers in the 503rd complete cycle of the pattern.

Therefore, 8^{2011} ends with the digit 2.

3.

Since $ABCD$ is a rectangle, $BC = AD = 12$.
Then $BE = BC + CE = 12 + 1 = 13$.

Since $BEDF$ is a quarter circle with centre B, $BF = BD = BE = 13$.

Using the Pythagorean Theorem in right $\triangle BCD$,
$DC^2 = DB^2 - BC^2 = 13^2 - 12^2 = 169 - 144 = 25$ and $DC = 5$ (since $DC > 0$).

Since $ABCD$ is a rectangle, $AB = DC = 5$.

Then $AF = BF - AB = 13 - 5 = 8$ cm.

Therefore, the length of AF is 8 cm.

4.

Let the three numbers be represented by a, b, and c.

Since the product of the first and second numbers equals the third number, $a \times b = c$. We are looking for $a \times b \times c = (a \times b) \times c = (c) \times c = c^2$. So when we find c^2 we have found the required product $a \times b \times c$.

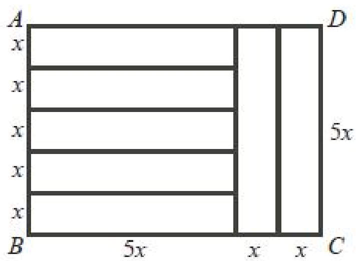
We know that $b \times c = 180$ and $b = 5 \times c$ so $b \times c = 180$ becomes $(5 \times c) \times c = 180$ or $5 \times c^2 = 180$. Dividing by 5, we obtain $c^2 = 36$. This is exactly what we are looking for since $a \times b \times c = c^2$.

Therefore, the product of the three numbers is 36.

For those who need to know what the actual numbers are, we can proceed and find the three numbers. We know $c^2 = 36$, so $c = 6$ since c is a positive number. So $b = 5 \times c = 5 \times (6) = 30$. And finally, $a \times b = c$ so $a \times (30) = 6$. Dividing by 30, we get $a = \frac{6}{30} = \frac{1}{5} = 0.2$. We can verify the product $a \times b \times c = (0.2) \times (30) \times (6) = 6 \times 6 = 36$.

5.

Let x be the width of one of the smaller identical rectangles, in cm. Five of the smaller rectangles are stacked on top of each other creating AB, so $AB = x + x + x + x + x = 5x$. Since $ABCD$ is a rectangle, $AB = CD = 5x$. But CD is the length of the smaller rectangle. Therefore, the smaller rectangle is $5x$ cm by x cm.



The area of rectangle $ABCD$ is the same as 7 times the area of one of the smaller rectangles.

$$\begin{aligned} \text{Area } ABCD &= 7 \times \text{Area of one smaller rectangle} \\ 560 &= 7 \times 5x \times x \\ 560 &= 35 \times x^2 \end{aligned}$$

Dividing both sides by 35, we obtain $x^2 = 16$ and $x = 4$ follows. ($x > 0$ since x is the width of the smaller rectangle.)

The width of the smaller rectangle is $x = 4$ cm and the length of the smaller rectangle is $5x = 5(4) = 20$ cm.

Therefore, the smaller rectangle is 20 cm long and 4 cm wide.