SECOND INTRA SCHOOL MATHEMATICS OLYMPIAD
ANSWER KEY
CLASS XI

1

Since m and n are consecutive positive integers with $n^2 - m^2 > 20$, then n is greater than m.
Therefore, we can write $n = m + 1$.
Since $n^2 - m^2 > 20$, then $(m+1)^2 - m^2 > 20$ or $m^2 + 2m + 1 - m^2 > 20$ or $2m > 19$ or $m > \frac{19}{2}$.
Since m is a positive integer, then $m \geq 10$.
Thus, we want to find the minimum value of $n^2 + m^2 = (m+1)^2 + m^2 = 2m^2 + 2m + 1$ when $m \geq 10$.
This minimum will occur when $m = 10$ (since $2m^2 + 2m + 1$ increases with m when m is a positive integer).
Therefore, the minimum possible value is $2(10^2) + 2(10) + 1 = 221$.

ANSWER: (E)

2

There are 64 cubes to start.
If we look at the bottom layer of cubes, we see that there are 6 uncovered cubes, each of which is missing 3 cubes above it. These are the only cubes that are missing.
Thus, there are $6(3) = 18$ missing cubes, so there are $64 - 18 = 46$ cubes remaining.

ANSWER: (A)

3

Solution 1
Since $\sqrt{n^2 + n^2 + n^2 + n^2} = 64$, then $\sqrt{4n^2} = 64$ or $2n = 64$, since $n > 0$.
Thus, $n = 32$.

Solution 2
Since $\sqrt{n^2 + n^2 + n^2 + n^2} = 64$, then $\sqrt{4n^2} = 64$.
Thus, $4n^2 = 64^2 = 4096$, and so $n^2 = 1024$.
Since $n > 0$, then $n = \sqrt{1024} = 32$.

ANSWER: (D)

4

. Since x is an integer, then $x + 1$ is an integer.
Since $\dfrac{-6}{x+1}$ is to be integer, then $x + 1$ must be a divisor of -6.
Thus, there are 8 possible values for $x + 1$, namely $-6, -3, -2, -1, 1, 2, 3$, and 6.
This gives 8 possible values for x, namely $-7, -4, -3, -2, 0, 1, 2$, and 5.

ANSWER: (A)

5

Since the three numbers in each straight line must have a product of 3240 and must include 45, then the other two numbers in each line must have a product of $\frac{3240}{45} = 72$.
The possible pairs of positive integers are 1 and 72, 2 and 36, 3 and 24, 4 and 18, 6 and 12, and 8 and 9.
The sums of the numbers in these pairs are 73, 38, 27, 22, 18, and 17.
To maximize the sum of the eight numbers, we want to choose the pairs with the largest possible sums, so we choose the first four pairs.
Thus, the largest possible sum of the eight numbers is $73 + 38 + 27 + 22 = 160$.

ANSWER: (E)

6

The bottom left vertex of the triangle has coordinates $(0, 0)$, since $y = x$ (the line with positive slope) passes through the origin.
The bottom right vertex of the triangle corresponds with the x-intercept of the line $y = -2x + 3$, which we find by setting $y = 0$ to obtain $-2x + 3 = 0$ or $x = \frac{3}{2}$. Thus, the bottom right vertex is $(\frac{3}{2}, 0)$.
The top vertex is the point of intersection of the two lines, which we find by combining the equations of the two lines to get $x = -2x + 3$ or $3x = 3$ or $x = 1$.
Thus, this point of intersection is $(1, 1)$.
Therefore, the triangle has a base along the x-axis of length $\frac{3}{2}$ and a height of length 1 (the y-coordinate of the top vertex).
Thus, the area of the triangle is $\frac{1}{2}\left(\frac{3}{2}\right)(1) = \frac{3}{4}$.

ANSWER: (A)

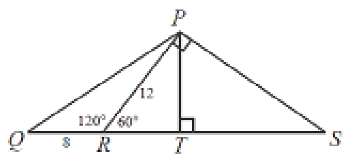
7

. Suppose that these two integers are x and $x+1$, since they are consecutive.
Then $(x+1)^2 - x^2 = 199$ or $(x^2+2x+1) - x^2 = 199$ or $2x+1 = 199$ or $x = 99$.
Therefore, the two integers are 99 and 100, and the sum of their squares is $99^2 + 100^2$ or $9801 + 10\,000 = 19\,801$.

ANSWER: (A)

8

Since $\angle QRP = 120^\circ$ and QRS is a straight line, then $\angle PRS = 180^\circ - 120^\circ = 60^\circ$.
Since $\angle RPS = 90^\circ$, then $\triangle SRP$ is a 30°-60°-90° triangle.
Therefore, $RS = 2PR = 2(12) = 24$.
Drop a perpendicular from P to T on RS.



Since $\angle PRT = 60^\circ$ and $\angle PTR = 90^\circ$, then $\triangle PRT$ is also a 30°-60°-90° triangle.
Therefore, $PT = \frac{\sqrt{3}}{2}PR = 6\sqrt{3}$.
Consider $\triangle QPS$. We may consider QS as its base with height PT.
Thus, its area is $\frac{1}{2}(6\sqrt{3})(8+24) = 96\sqrt{3}$.

ANSWER: (E)

9

. Suppose that Ivan ran a distance of x km on Monday.
Then on Tuesday, he ran $2x$ km, on Wednesday, he ran x km, on Thursday, he ran $\frac{1}{2}x$ km, and on Friday he ran x km.
The shortest of any of his runs was on Thursday, so $\frac{1}{2}x = 5$ or $x = 10$.
Therefore, his runs were 10 km, 20 km, 10 km, 5 km, and 10 km, for a total of 55 km.

ANSWER: (A)

.10

. *Solution 1*
Accounting for all of the missing numbers, $M + N + P + Q + R = 1 + 4 + 5 + 6 + 7 = 23$.
To determine the sum $M + N + P + Q$, we can determine the value of R and subtract this from 23.
R cannot be 1, since $0 + 1 = 1$ is not prime. (If R was 1, we would thus have the sum of the numbers at the ends of one of the edges not equal to a prime number.)
R cannot be 4, since $0 + 4 = 4$ is not prime.
R cannot be 6, since $0 + 6 = 6$ is not prime.
R cannot be 7, since $2 + 7 = 9$ is not prime.
By process of elimination, $R = 5$, so $M + N + P + Q = 23 - 5 = 18$.
(We will see in Solution 2 that we can fill in the rest of the numbers in a way that satisfies the requirements.)

Solution 2
The missing numbers are 1, 4, 5, 6, 7.
Since $Q + 3$ must be a prime number, then Q must be 4 (since $1 + 3$, $5 + 3$, $6 + 3$, and $7 + 3$ are not prime).
Since $M + 0$ and $M + 4$ must both be prime numbers, then M must be 7 (since $1 + 0$, $5 + 4$ and $6 + 4$ are not prime).
Since $P + 2$ and $P + 4$ must both be prime numbers, then P must be 1 (since $5 + 4$ and $6 + 2$ are not prime).
Since $N + 7$ and $N + 1$ must both be prime numbers, then N must be 6 (since $5 + 7$ is not prime).
We can check that if $R = 5$, then the requirement that the sum of the two numbers at the ends of each edge be a prime number is met.
Thus, $M + N + P + Q = 7 + 6 + 1 + 4 = 18$.

ANSWER: (C)

11

Solution 1

Since $AB = BC$, then B lies on the perpendicular bisector of AC.
Since A has coordinates $(2, 2)$ and C has coordinates $(8, 4)$, then the midpoint of AC is $\left(\frac{1}{2}(2+8), \frac{1}{2}(4+2)\right) = (5, 3)$ and the slope of AC is $\frac{4-2}{8-2} = \frac{1}{3}$.
Therefore, the slope of the perpendicular bisector is -3 (the negative reciprocal of $\frac{1}{3}$) and it passes through $(5, 3)$, so has equation $y - 3 = -3(x - 5)$ or $y = -3x + 18$.
The x-intercept of this line comes when y is set to 0; here, we obtain $x = 6$.
Therefore, since B is the point where the perpendicular bisector of AC crosses the x-axis, then the x-coordinate of B is 6.
(We can check that indeed if B has coordinates $(6, 0)$, then AB and BC are perpendicular.)

Solution 2

Since $\triangle ABC$ is an isosceles right-angled triangle, then $\angle ABC = 90^\circ$, and so AB is perpendicular to BC.
Suppose B has coordinates $(b, 0)$.
The slope of AB is $\frac{2-0}{2-b}$ and the slope of BC is $\frac{4-0}{8-b}$.
Since AB and BC are perpendicular, their slopes are negative reciprocals, so

$$\begin{aligned} \frac{2}{2-b} &= -\frac{8-b}{4} \\ -8 &= (2-b)(8-b) \\ -8 &= b^2 - 10b + 16 \\ b^2 - 10b + 24 &= 0 \\ (b-4)(b-6) &= 0 \end{aligned}$$

and so $b = 4$ or $b = 6$.
We must determine which value of b gives $AB = BC$ (since we have already used the perpendicularity).
If $b = 4$, then $AB = \sqrt{(4-2)^2 + (0-2)^2} = \sqrt{8}$ and $BC = \sqrt{(8-4)^2 + (4-0)^2} = \sqrt{32}$ and so $AB \neq BC$.
Therefore, the x-coordinate of B must be 6.
(We can check that, in this case, AB does equal BC.)

Solution 3

To go from A to C, we go 6 units right and 2 units up.
Suppose that to go from A to B, we go p units right and q units down, were $p, q > 0$.
Since BC is equal and perpendicular to AB, then to go from B to C, we must go q units right and p units up.
(We can see this by looking at the slopes of segments AB and BC.)
Therefore, to get from A to C through B, we go $p+q$ units right and $q-p$ units up, so $p+q = 6$ and $q - p = 2$, as the result is the same as from going directly to C from A.
Since $p + q = 6$ and $q - p = 2$, then $2q = 8$ (adding the equations), so $q = 4$, and so $p = 2$.
Since A has coordinates $(2, 2)$, then B has coordinates $(6, 0)$ which lies on the x-axis as required.

ANSWER: (D)

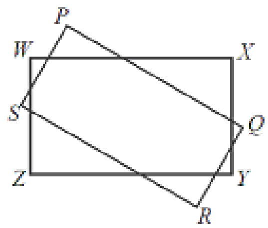
12

Since each term after the third is the sum of the preceding three terms, then, looking at the fourth term, $13 = 5 + p + q$ or $p + q = 8$.
Looking at the fifth term, $r = p + q + 13 = 8 + 13 = 21$.
Looking at the seventh term, $x = 13 + r + 40 = 13 + 21 + 40 = 74$.

ANSWER: (D)

13

Consider rectangles $WXYZ$ and $PQRS$.
Each of the four sides of $PQRS$ can intersect at most 2 of the sides of $WXYZ$, as any straight line can intersect at most two sides of a rectangle.
Therefore, the maximum possible number of points of intersection between the two rectangles is 8.
8 points of intersection is possible, as we can see in the diagram:

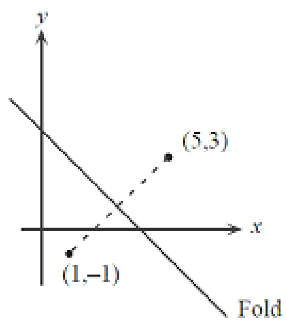


So the maximum possible number of points of intersection is 8.

Answer: (D)

14

. Since the point $(5,3)$ lands on the point $(1,-1)$ when folded, then the fold line must pass through the midpoint of these two points, namely $\left(\frac{1}{2}(5+1), \frac{1}{2}(3+(-1))\right) = (3,1)$.



Of the given possibilities, $(3,1)$ lies only on the line $y = -x + 4$, so (D) is the answer.
(In fact, the fold line must be the perpendicular bisector of the line segment through $(5,3)$ and $(1,-1)$. The slope of the line segment through $(5,3)$ and $(1,-1)$ is $\frac{3-(-1)}{5-1} = 1$, so the perpendicular bisector has slope -1.
Since the perpendicular bisector has slope -1 and passes through $(3,1)$, then it has equation $y = -x + 4$.)

Answer: (D)

15

Between 5000 and 6000, every integer except for 6000 has thousands digit equal to 5.
Note that the number 6000 does not have the desired property.
Thus, we are looking for integers $5xyz$ with $x + y + z = 5$.
The possible combinations of three digits for x, y and z are: 5, 0, 0; 4, 1, 0; 3, 2, 0; 3, 1, 1; 2, 2, 1.
A combination of three different digits (like 4, 1, 0) can be arranged in 6 ways: 410, 401, 140, 104, 041, 014.
A combination of three digits with one repeated (like 5, 0, 0) can be arranged in 3 ways: 500, 050, 005.
Therefore, 5, 0, 0 and 3, 1, 1 and 2, 2, 1 each give 3 integers, and 4, 1, 0 and 3, 2, 0 each give 6 integers.
So the number of integers with the desired property is $3(3) + 2(6) = 21$.

Answer: (C)

16

. Since $\dfrac{p+q^{-1}}{p^{-1}+q}=17$, then $\dfrac{p+\frac{1}{q}}{\frac{1}{p}+q}=17$ or $\dfrac{\frac{pq+1}{q}}{\frac{1+pq}{p}}=17$ or $\dfrac{p(pq+1)}{q(pq+1)}=17$.

Since p and q are positive integers, then $pq+1>0$, so we can divide out the common factor in the numerator and denominator to obtain $\dfrac{p}{q}=17$ or $p=17q$.
Since p and q are positive integers, then $q\geq 1$.
Since $p+q\leq 100$, then $17q+q\leq 100$ or $18q\leq 100$ or $q\leq \frac{100}{18}=5\frac{5}{9}$.
Since q is a positive integer, then $q\leq 5$.
Therefore, the combined restriction is $1\leq q\leq 5$, and so there are five pairs.
(We can check that these pairs are $(p,q)=(17,1),(34,2),(51,3),(68,4),(85,5)$.)

ANSWER: (E)

17

Since the circle has radius 1, then its area is $\pi(1^2)=\pi$.
Since the square and the circle have the same area, then the side length of the square is $\sqrt{\pi}$.
Let M be the midpoint of line segment PQ.
Since PQ is a chord of the circle, then OM is perpendicular to line segment PQ.
Since OM is perpendicular to PQ and O is the centre of the square, then OM is half of the length of one of the sides of the square, so $OM=\frac{1}{2}\sqrt{\pi}$.
By the Pythagorean Theorem in $\triangle OPM$, we have $PM^2=OP^2-OM^2=1^2-\left(\frac{1}{2}\sqrt{\pi}\right)^2=1-\frac{1}{4}\pi$.
Therefore, $PQ=2PM=2\sqrt{1-\frac{1}{4}\pi}=\sqrt{4(1-\frac{1}{4}\pi)}=\sqrt{4-\pi}$.

ANSWER: (A)

18

Adding the second and third equations, we obtain

$$\begin{aligned} ac+bd+ad+bc &= 77 \\ ac+ad+bc+bd &= 77 \\ a(c+d)+b(c+d) &= 77 \\ (a+b)(c+d) &= 77 \end{aligned}$$

Since each of a, b, c and d is a positive integer, then $a+b$ and $c+d$ are each positive integers and are each at least 2.
Since the product of $a+b$ and $c+d$ is $77=7\times 11$ (with 7 and 11 both prime), then one must equal 7 and the other must equal 11.
Therefore, $a+b+c+d=7+11=18$.
(We can check with some work that $(a,b,c,d)=(5,2,4,7)$ is a solution to the system.)

ANSWER: (D)

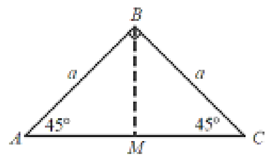
19

. Since each term is obtained by adding the same number to the previous term, then the differences between pairs of consecutive terms are equal.
Looking at the first three terms, we thus have $2a-a=b-2a$ or $b=3a$.
Therefore, in terms of a, the first four terms are a, $2a$, $3a$, and $a-6-3a=-6-2a$.
Since the constant difference between the terms equals a (as $2a-a=a$), then the fourth term should be $4a$, so $4a=-6-2a$ or $6a=-6$ or $a=-1$.
Thus, the sequence begins -1, -2, -3, -4.
The 100th term is thus -100 (which we can get by inspection or by saying that we must add the common difference 99 times to the first term, to get $-1+99(-1)=-100$).

ANSWER: (A)

20

We label the stages in this process as Stage 0 (a square), Stage 1 (2 triangles), Stage 2 (4 triangles), Stage 3 (8 triangles), and Stage 4 (16 triangles).
We want to determine the length of the longest edge of one of the 16 triangles in Stage 4.
At Stage 1, we have two right-angled isosceles triangles with legs of length 4.
Consider a general right-angled isosceles triangle ABC with legs AB and BC of length a.
Since this is a 45°-45°-90° triangle, its hypotenuse AC has length $\sqrt{2}$a.
We split the triangle into two equal pieces by bisecting the right-angle at B:



Since $\triangle ABC$ is isosceles, then this bisecting line is both an altitude and a median. In other words, it is perpendicular to AC at M and M is the midpoint of AC.
Therefore, the two triangular pieces $\triangle AMB$ and $\triangle CMB$ are identical 45°-45°-90° triangles.
The longest edges of these triangles (AB and CB) are the legs of the original triangle, and so have length a.
Since the longest edge of the original triangle was $\sqrt{2}a$, then the longest edge has been reduced by a factor of $\sqrt{2}$.
Since we have shown that this is the case for an arbitrary isosceles right-angled triangle, we can then apply this property to our problem.
In Stage 1, the longest edge has length $4\sqrt{2}$.