CLASS XII CHAPTER 1 RELATION AND FUNCTION

MISC. EX. SOLUTIONS

1. Let $f: \mathbf{R} \to \mathbf{R}$ be defined as $f(x) = 10x + 7$. Find the function $g: \mathbf{R} \to \mathbf{R}$ such that $g \circ f = f \circ g = 1_{\mathbf{R}}$.

ANS:

It is given that $f: \mathbf{R} \to \mathbf{R}$ is defined as $f(x) = 10x + 7$.

One-one:

Let $f(x) = f(y)$, where $x, y \in \mathbf{R}$.

$\Rightarrow 10x + 7 = 10y + 7$

$\Rightarrow x = y$

$\therefore f$ is a one-one function.

Onto:

For $y \in \mathbf{R}$, let $y = 10x + 7$.

$\Rightarrow x = \frac{y-7}{10} \in \mathbf{R}$

Therefore, for any $y \in \mathbf{R}$, there exists $x = \frac{y-7}{10} \in \mathbf{R}$ such that $f(x) = f\left(\frac{y-7}{10}\right) = 10\left(\frac{y-7}{10}\right) + 7 = y - 7 + 7 = y$.

$\therefore f$ is onto.

Therefore, f is one-one and onto.

Thus, f is an invertible function.

Let us define $g: \mathbf{R} \to \mathbf{R}$ as $g(y) = \frac{y-7}{10}$.

Now, we have:

$$gof(x) = g(f(x)) = g(10x+7) = \frac{(10x+7)-7}{10} = \frac{10x}{10} = 10$$

And,

$$fog(y) = f(g(y)) = f\left(\frac{y-7}{10}\right) = 10\left(\frac{y-7}{10}\right) + 7 = y - 7 + 7 = y$$

$\therefore gof = I_{\mathbf{R}}$ and $fog = I_{\mathbf{R}}$

Hence, the required function $g: \mathbf{R} \to \mathbf{R}$ is defined as $g(y) = \frac{y-7}{10}$.

2. Let f : W → W be defined as $f(n) = n - 1$, if n is odd and $f(n) = n + 1$, if n is even. Show that f is invertible. Find the inverse of f. Here, W is the set of all whole numbers.

ANS:

It is given that:

f: W → W is defined as $f(n)=\begin{cases} n-1, \text{ if } n \text{ is odd} \\ n+1, \text{ if } n \text{ is even} \end{cases}$

One-one:

Let $f(n) = f(m)$.

It can be observed that if n is odd and m is even, then we will have $n - 1 = m + 1$.

$\Rightarrow n - m = 2$

However, this is impossible.

Similarly, the possibility of n being even and m being odd can also be ignored under a similar argument.

∴Both n and m must be either odd or even.

Now, if both n and m are odd, then we have:

$f(n) = f(m) \Rightarrow n - 1 = m - 1 \Rightarrow n = m$

Again, if both n and m are even, then we have:

$f(n) = f(m) \Rightarrow n + 1 = m + 1 \Rightarrow n = m$

∴ f is one-one.

It is clear that any odd number $2r + 1$ in co-domain N is the image of $2r$ in domain N and any even number $2r$ in co-domain N is the image of $2r + 1$ in domain N.

∴ f is onto.

Hence, f is an invertible function.

Let us define g: W → W as:

$$g(m)=\begin{cases} m+1, \text{ if } m \text{ is even} \\ m-1, \text{ if } m \text{ is odd} \end{cases}$$

Now, when n is odd:

$$gof(n)=g(f(n))=g(n-1)=n-1+1=n$$

And, when n is even:

$$gof(n)=g(f(n))=g(n+1)=n+1-1=n$$

Similarly, when m is odd:

$$fog(m)=f(g(m))=f(m-1)=m-1+1=m$$

When m is even:

$$fog(m)=f(g(m))=f(m+1)=m+1-1=m$$

$\therefore \therefore gof=I_W$ and $fog=I_W$

Thus, f is invertible and the inverse of f is given by $f^{-1}=g$, which is the same as f.

Hence, the inverse of f is f itself.

3. **If f : R → R is defined by $f(x) = x^2 - 3x + 2$, find $f(f(x))$.**

ANS:

It is given that f: R → R is defined as $f(x) = x^2 - 3x + 2$.

$$\begin{aligned} f(f(x)) &= f(x^2-3x+2) \\ &= (x^2-3x+2)^2-3(x^2-3x+2)+2 \\ &= x^4+9x^2+4-6x^3-12x+4x^2-3x^2+9x-6+2 \\ &= x^4-6x^3+10x^2-3x \end{aligned}$$

4. **Show that the function $f: \mathbf{R} \to \{x \in \mathbf{R} : -1 < x < 1\}$ defined by $f(x) = \dfrac{x}{1+|x|}$, $x \in \mathbf{R}$ is one one and onto function.**

It is given that $f: \mathbf{R} \to \{x \in \mathbf{R}: -1 < x < 1\}$ is defined as $f(x) = \dfrac{x}{1+|x|}$, $x \in \mathbf{R}$.

Suppose $f(x) = f(y)$, where $x, y \in \mathbf{R}$.

$$\Rightarrow \frac{x}{1+|x|} = \frac{y}{1+|y|}$$

It can be observed that if x is positive and y is negative, then we have:

$$\frac{x}{1+x} = \frac{y}{1-y} \Rightarrow 2xy = x - y$$

Since x is positive and y is negative:

$x > y \Rightarrow x - y > 0$

But, $2xy$ is negative.

Then, $2xy \neq x - y$.

Thus, the case of x being positive and y being negative can be ruled out.

Under a similar argument, x being negative and y being positive can also be ruled out

$\therefore$ x and y have to be either positive or negative.

When x and y are both positive, we have:

$$f(x) = f(y) \Rightarrow \frac{x}{1+x} = \frac{y}{1+y} \Rightarrow x + xy = y + xy \Rightarrow x = y$$

When x and y are both negative, we have:

$$f(x)=f(y)\Rightarrow \frac{x}{1-x}=\frac{y}{1-y}\Rightarrow x-xy=y-yx\Rightarrow x=y$$

$\therefore f$ is one-one.

Now, let $y \in \mathbf{R}$ such that $-1 < y < 1$.

If y is negative, then there exists $x=\frac{y}{1+y}\in \mathbf{R}$ such that

$$f(x)=f\left(\frac{y}{1+y}\right)=\frac{\left(\frac{y}{1+y}\right)}{1+\left|\frac{y}{1+y}\right|}=\frac{\frac{y}{1+y}}{1+\left(\frac{-y}{1+y}\right)}=\frac{y}{1+y-y}=y.$$

If y is positive, then there exists $x=\frac{y}{1-y}\in \mathbf{R}$ such that

$$f(x)=f\left(\frac{y}{1-y}\right)=\frac{\left(\frac{y}{1-y}\right)}{1+\left|\left(\frac{y}{1-y}\right)\right|}=\frac{\frac{y}{1-y}}{1+\frac{y}{1-y}}=\frac{y}{1-y+y}=y.$$

$\therefore f$ is onto.

Hence, f is one-one and onto.

5. Show that the function $f: \mathbf{R}\rightarrow \mathbf{R}$ given by $f(x)=x^3$ is injective.

ANS:

$f: \mathbf{R}\rightarrow \mathbf{R}$ is given as $f(x)=x^3$.

Suppose $f(x)=f(y)$, where $x, y \in \mathbf{R}$.

$\Rightarrow x^3 = y^3$ … (1)

Now, we need to show that $x = y$.

Suppose $x \neq y$, their cubes will also not be equal.

$\Rightarrow x^3 \neq y^3$

However, this will be a contradiction to (1).

$\therefore x = y$

Hence, f is injective.

6. Give examples of two functions $f: \mathbf{N} \to \mathbf{Z}$ and $g: \mathbf{Z} \to \mathbf{Z}$ such that $g \circ f$ is injective but g is not injective.

(Hint : Consider $f(x) = x$ and $g(x) = |x|$).

ANS:

Define $f: \mathrm{N} \to \mathrm{Z}$ as $f(x) = x$ and $g: \mathrm{Z} \to \mathrm{Z}$ as $g(x) = |x|$.

We first show that g is not injective.

It can be observed that:

$g(-1) = |-1| = 1$

$g(1) = |1| = 1$

$\therefore g(-1) = g(1)$, but $-1 \neq 1$.

$\therefore g$ is not injective.

Now, $gof: \mathrm{N} \to \mathrm{Z}$ is defined as $gof(x) = g(f(x)) = g(x) = |x|$.

Let $x, y \in \mathrm{N}$ such that $gof(x) = gof(y)$.

$\Rightarrow |x| = |y|$

Since x and $y \in \mathrm{N}$, both are positive.

$\therefore |x| = |y| \Rightarrow x = y$

Hence, gof is injective

7. Give examples of two functions $f: \mathbf{N} \to \mathbf{N}$ and $g: \mathbf{N} \to \mathbf{N}$ such that $g \circ f$ is onto but f is not onto.

(Hint : Consider $f(x) = x + 1$ and $g(x) = \begin{cases} x-1 \text{ if } x > 1 \\ 1 \quad \text{if } x = 1 \end{cases}$

ANS:

Define f: N → N by,

$f(x) = x + 1$

And, g: N → N by,

$$g(x)=\begin{cases}x-1 & \text{if } x>1\\ 1 & \text{if } x=1\end{cases}$$

We first show that g is not onto.

For this, consider element 1 in co-domain N. It is clear that this element is not an image of any of the elements in domain N.

∴ f is not onto.

Now, gof: N → N is defined by,

$$gof(x)=g(f(x))=g(x+1)=(x+1)-1 \quad [x\in \mathbf{N}\Rightarrow (x+1)>1]$$
$$= x$$

Then, it is clear that for $y \in$ N, there exists $x = y \in$ N such that $gof(x) = y$.

Hence, gof is onto.

8. Given a non empty set X, consider P(X) which is the set of all subsets of X.

Define the relation R in P(X) as follows:
For subsets A, B in P(X), ARB if and only if $A \subset B$. Is R an equivalence relation on P(X)? Justify your answer.

ANS:

Since every set is a subset of itself, ARA for all $A \in P(X)$.

$\therefore$R is reflexive.

Let $ARB \Rightarrow A \subset B$.

This cannot be implied to $B \subset A$.

For instance, if $A = \{1, 2\}$ and $B = \{1, 2, 3\}$, then it cannot be implied that B is related to A.

$\therefore$ R is not symmetric.

Further, if ARB and BRC, then $A \subset B$ and $B \subset C$.

$\Rightarrow A \subset C$

$\Rightarrow ARC$

$\therefore$ R is transitive.

Hence, R is not an equivalence relation since it is not symmetric.

9. **Given a non-empty set X, consider the binary operation $* : P(X) \times P(X) \to P(X)$ given by $A * B = A \cap B \ \forall A, B$ in P(X), where P(X) is the power set of X. Show that X is the identity element for this operation and X is the only invertible element in P(X) with respect to the operation $*$.**

ANS:

It is given that $* : P(X) \times P(X) \to P(X)$ is defined as $A * B = A \cap B \ \forall A, B \in P(X)$.

We know that $A \cap X = A = X \cap A \ \forall A \in P(X)$.

$\Rightarrow A * X = A = X * A \ \forall A \in P(X)$

Thus, X is the identity element for the given binary operation *.

Now, an element $A \in P(X)$ is invertible if there exists $B \in P(X)$ such that

$A * B = X = B * A$. (As X is the identity element)

i.e.,

$A \cap B = X = B \cap A$

This case is possible only when $A = X = B$.

Thus, X is the only invertible element in $P(X)$ with respect to the given operation*.

Hence, the given result is proved.

10. Find the number of all onto functions from the set $\{1, 2, 3, \ldots, n\}$ to itself.

ANS:

Onto functions from the set $\{1, 2, 3, \ldots, n\}$ to itself is simply a permutation on n symbols $1, 2, \ldots, n$.

Thus, the total number of onto maps from $\{1, 2, \ldots, n\}$ to itself is the same as the total number of permutations on n symbols $1, 2, \ldots, n$, which is n.

11. Let $S = \{a, b, c\}$ and $T = \{1, 2, 3\}$. Find F^{-1} of the following functions F from S to T, if it exists.

(i) $F = \{(a, 3), (b, 2), (c, 1)\}$ (ii) $F = \{(a, 2), (b, 1), (c, 1)\}$

ANS:

$S = \{a, b, c\}$, $T = \{1, 2, 3\}$

(i) F: $S \to T$ is defined as:

$F = \{(a, 3), (b, 2), (c, 1)\}$

$\Rightarrow F(a) = 3, F(b) = 2, F(c) = 1$

Therefore, $F^{-1}: T \to S$ is given by

$F^{-1} = \{(3, a), (2, b), (1, c)\}$.

(ii) F: $S \to T$ is defined as:

$F = \{(a, 2), (b, 1), (c, 1)\}$

Since $F(b) = F(c) = 1$, F is not one-one.

Hence, F is not invertible i.e., F^{-1} does not exist.

12. Consider the binary operations $* : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ and $o : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ defined as $a * b = |a - b|$ and $a \text{ o } b = a$, $\forall a, b \in \mathbf{R}$. Show that $*$ is commutative but not associative, o is associative but not commutative. Further, show that $\forall a, b, c \in \mathbf{R}$, $a * (b \text{ o } c) = (a * b) \text{ o } (a * b)$. [If it is so, we say that the operation $*$ distributes over the operation o]. Does o distribute over $*$? Justify your answer.

ANS:

It is given that $*: \mathbf{R} \times \mathbf{R} \to$ and $\circ: \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ isdefined as

$a * b = |a - b|$ and $a \circ b = a, \forall a, b \in \mathbf{R}$.

For $a, b \in \mathbf{R}$, we have:

$a * b = |a - b|$

$b * a = |b - a| = |-(a - b)| = |a - b|$

$\therefore a * b = b * a$

$\therefore$ The operation * is commutative.

It can be observed that,

$(1 * 2) * 3 = (|1 - 2|) * 3 = 1 * 3 = |1 - 3| = 2$
$1 * (2 * 3) = 1 * (|2 - 3|) = 1 * 1 = |1 - 1| = 0$
$\therefore (1 * 2) * 3 \neq 1 * (2 * 3)$ (where $1, 2, 3 \in \mathbf{R}$)

$\therefore$The operation * is not associative.

Now, consider the operation o:

It can be observed that $1 \circ 2 = 1$ and $2 \circ 1 = 2$.

$\therefore 1 \circ 2 \neq 2 \circ 1$ (where $1, 2 \in \mathbf{R}$)

$\therefore$The operation o is not commutative.

Let $a, b, c \in \mathbf{R}$. Then, we have:

$(a \circ b) \circ c = a \circ c = a$

$a \circ (b \circ c) = a \circ b = a$

$\Rightarrow a \circ b) \circ c = a \circ (b \circ c)$

$\therefore$ The operation o is associative.

Now, let $a, b, c \in \mathbb{R}$, then we have:

$a * (b \circ c) = a * b = |a - b|$

$(a * b) \circ (a * c) = (|a - b|) \circ (|a - c|) = |a - b|$

Hence, $a * (b \circ c) = (a * b) \circ (a * c)$.

Now,

$1 \circ (2 * 3) = 1 \circ (|2 - 3|) = 1 \circ 1 = 1$

$(1 \circ 2) * (1 \circ 3) = 1 * 1 = |1 - 1| = 0$

$\therefore 1 \circ (2 * 3) \neq (1 \circ 2) * (1 \circ 3)$ (where $1, 2, 3 \in \mathbb{R}$)

$\therefore$ The operation o does not distribute over *.

13. Given a non-empty set X, let $* : P(X) \times P(X) \rightarrow P(X)$ be defined as $A * B = (A - B) \cup (B - A), \forall A, B \in P(X)$. Show that the empty set ϕ is the identity for the operation $*$ and all the elements A of P(X) are invertible with $A^{-1} = A$. (Hint : $(A - \phi) \cup (\phi - A) = A$ and $(A - A) \cup (A - A) = A * A = \phi$).

ANS:

It is given that $*: P(X) \times P(X) \to P(X)$ is defined as

$A * B = (A - B) \cup (B - A)\ \forall\ A, B \in P(X)$.

Let $A \in P(X)$. Then, we have:

$A * \Phi = (A - \Phi) \cup (\Phi - A) = A \cup \Phi = A$

$\Phi * A = (\Phi - A) \cup (A - \Phi) = \Phi \cup A = A$

$\therefore A * \Phi = A = \Phi * A.\ \forall\ A \in P(X)$

Thus, Φ is the identity element for the given operation*.

Now, an element $A \in P(X)$ will be invertible if there exists $B \in P(X)$ such that

$A * B = \Phi = B * A$. (As Φ is the identity element)

Now, we observed that $A * A = (A - A) \cup (A - A) = \phi \cup \phi = \phi\ \forall A \in P(X)$.

Hence, all the elements A of $P(X)$ are invertible with $A^{-1} = A$.

14. Define a binary operation $*$ on the set $\{0, 1, 2, 3, 4, 5\}$ as

$$a * b = \begin{cases} a + b, & \text{if } a + b < 6 \\ a + b - 6 & \text{if } a + b \geq 6 \end{cases}$$

Show that zero is the identity for this operation and each element a of the set is invertible with $6 - a$ being the inverse of a.

ANS:

Let $X = \{0, 1, 2, 3, 4, 5\}$.

The operation $*$ on X is defined as:

$$a * b = \begin{cases} a + b & \text{if } a + b < 6 \\ a + b - 6 & \text{if } a + b \geq 6 \end{cases}$$

An element $e \in X$ is the identity element for the operation $*$, if $a * e = a = e * a\ \forall a \in X$.

For $a \in X$, we observed that:

$a * 0 = a + 0 = a \qquad [a \in X \Rightarrow a + 0 < 6]$

$0 * a = 0 + a = a \qquad [a \in X \Rightarrow 0 + a < 6]$

$\therefore a * 0 = a = 0 * a\ \forall a \in X$

Thus, 0 is the identity element for the given operation $*$.

An element $a \in X$ is invertible if there exists $b \in X$ such that $a * b = 0 = b * a$.

$$\text{i.e., } \begin{cases} a + b = 0 = b + a, & \text{if } a + b < 6 \\ a + b - 6 = 0 = b + a - 6, & \text{if } a + b \geq 6 \end{cases}$$

i.e.,

$a = -b$ or $b = 6 - a$

But, $X = \{0, 1, 2, 3, 4, 5\}$ and $a, b \in X$. Then, $a \neq -b$.

$\therefore b = 6 - a$ is the inverse of a $\forall$ $a \in X$.

Hence, the inverse of an element $a \in X$, $a \neq 0$ is $6 - a$ i.e., $a^{-1} = 6 - a$.

15. Let A = {– 1, 0, 1, 2}, B = {– 4, – 2, 0, 2} and f, g : A → B be functions defined by $f(x) = x^2 - x$, $x \in$ A and $g(x) = 2\left|x - \frac{1}{2}\right| - 1$, $x \in$ A. Are f and g equal? Justify your answer. (Hint: One may note that two functions f : A → B and g : A → B such that $f(a) = g(a)$ $\forall$ $a \in$ A, are called equal functions).

ANS:

It is given that $A = \{-1, 0, 1, 2\}$, $B = \{-4, -2, 0, 2\}$.

Also, it is given that $f, g: A \rightarrow B$ are defined by $f(x) = x^2 - x$, $x \in A$ and $g(x) = 2\left|x - \frac{1}{2}\right| - 1$, $x \in A$.

It is observed that:

$$f(-1) = (-1)^2 - (-1) = 1 + 1 = 2$$
$$g(-1) = 2\left|(-1) - \frac{1}{2}\right| - 1 = 2\left(\frac{3}{2}\right) - 1 = 3 - 1 = 2$$
$$\Rightarrow f(-1) = g(-1)$$
$$f(0) = (0)^2 - 0 = 0$$
$$g(0) = 2\left|0 - \frac{1}{2}\right| - 1 = 2\left(\frac{1}{2}\right) - 1 = 1 - 1 = 0$$
$$\Rightarrow f(0) = g(0)$$
$$f(1) = (1)^2 - 1 = 1 - 1 = 0$$
$$g(1) = 2\left|1 - \frac{1}{2}\right| - 1 = 2\left(\frac{1}{2}\right) - 1 = 1 - 1 = 0$$
$$\Rightarrow f(1) = g(1)$$
$$f(2) = (2)^2 - 2 = 4 - 2 = 2$$
$$g(2) = 2\left|2 - \frac{1}{2}\right| - 1 = 2\left(\frac{3}{2}\right) - 1 = 3 - 1 = 2$$
$$\Rightarrow f(2) = g(2)$$
$$\therefore f(a) = g(a) \ \forall a \in A$$

Hence, the functions f and g are equal.

16. Let $A = \{1, 2, 3\}$. Then number of relations containing $(1, 2)$ and $(1, 3)$ which are reflexive and symmetric but not transitive is

(A) 1 (B) 2 (C) 3 (D) 4

ANS:

The given set is $A = \{1, 2, 3\}$.

The smallest relation containing (1, 2) and (1, 3) which is reflexive and symmetric, but not transitive is given by:

$R = \{(1, 1), (2, 2), (3, 3), (1, 2), (1, 3), (2, 1), (3, 1)\}$

This is because relation R is reflexive as $(1, 1), (2, 2), (3, 3) \in R$.

Relation R is symmetric since $(1, 2), (2, 1) \in R$ and $(1, 3), (3, 1) \in R$.

But relation R is not transitive as $(3, 1), (1, 2) \in R$, but $(3, 2) \notin R$.

Now, if we add any two pairs (3, 2) and (2, 3) (or both) to relation R, then relation R will become transitive.

Hence, the total number of desired relations is one.

The correct answer is A.

17. Let $A = \{1, 2, 3\}$. Then number of equivalence relations containing $(1, 2)$ is

(A) 1 (B) 2 (C) 3 (D) 4

ANS:

It is given that $A = \{1, 2, 3\}$.

The smallest equivalence relation containing (1, 2) is given by,

$R_1 = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\}$

Now, we are left with only four pairs i.e., (2, 3), (3, 2), (1, 3), and (3, 1).

If we odd any one pair [say (2, 3)] to R_1, then for symmetry we must add (3, 2). Also, for transitivity we are required to add (1, 3) and (3, 1).

Hence, the only equivalence relation (bigger than R_1) is the universal relation.

This shows that the total number of equivalence relations containing (1, 2) is two.

The correct answer is B.

18. Let $f : \mathbf{R} \to \mathbf{R}$ be the Signum Function defined as

$$f(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$

and $g : \mathrm{R} \to \mathrm{R}$ be the Greatest Integer Function given by $g(x) = [x]$, where $[x]$ is greatest integer less than or equal to x. Then, does fog and gof coincide in (0, 1]?

ANS:

It is given that,

$f : \mathbf{R} \to \mathbf{R}$ is defined as $f(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$

Also, $g : \mathbf{R} \to \mathbf{R}$ is defined as $g(x) = [x]$, where $[x]$ is the greatest integer less than or equal to x.

Now, let $x \in (0, 1]$.

Then, we have:

$[x] = 1$ if $x = 1$ and $[x] = 0$ if $0 < x < 1$.

$$\therefore fog(x) = f(g(x)) = f([x]) = \begin{cases} f(1), & \text{if } x = 1 \\ f(0), & \text{if } x \in (0,1) \end{cases} = \begin{cases} 1, & \text{if } x = 1 \\ 0, & \text{if } x \in (0,1) \end{cases}$$

$$\begin{aligned} gof(x) &= g(f(x)) \\ &= g(1) \qquad [x > 0] \\ &= [1] = 1 \end{aligned}$$

Thus, when $x \in (0, 1)$, we have $fog(x) = 0$ and $gof(x) = 1$.

Hence, fog and gof do not coincide in (0, 1].

19. Number of binary operations on the set $\{a, b\}$ are

(A) 10 (B) 16 (C) 20 (D) 8

ANS:

A binary operation * on $\{a, b\}$ is a function from $\{a, b\} \times \{a, b\} \to \{a, b\}$

i.e., * is a function from $\{(a, a), (a, b), (b, a), (b, b)\} \to \{a, b\}$.

Hence, the total number of binary operations on the set $\{a, b\}$ is 2^4 i.e., 16.

The correct answer is B.