

## CLASS XII

### CHAPTER 5 CONTINUITY AND DIFFERENTIABILITY

#### NCERT EX 5.8 SOLUTIONS

##### Question 1:

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Verify Rolle's Theorem for the function  $f(x) = x^2 + 2x - 8$ ,  $x \in [-4, 2]$

##### ANS:

The given function,  $f(x) = x^2 + 2x - 8$ , being a polynomial function, is continuous in  $[-4, 2]$  and is differentiable in  $(-4, 2)$ .

$$f(-4) = (-4)^2 + 2 \times (-4) - 8 = 16 - 8 - 8 = 0$$

$$f(2) = (2)^2 + 2 \times 2 - 8 = 4 + 4 - 8 = 0$$

$$\therefore f(-4) = f(2) = 0$$

$\Rightarrow$  The value of  $f(x)$  at  $-4$  and  $2$  coincides.

Rolle's Theorem states that there is a point  $c \in (-4, 2)$  such that  $f'(c) = 0$

$$f(x) = x^2 + 2x - 8$$

$$\Rightarrow f'(x) = 2x + 2$$

$$\therefore f'(c) = 0$$

$$\Rightarrow 2c + 2 = 0$$

$$\Rightarrow c = -1, \text{ where } c = -1 \in (-4, 2)$$

Hence, Rolle's Theorem is verified for the given function.

##### Question 2:

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Examine if Rolle's Theorem is applicable to any of the following functions. Can you say some thing about the converse of Rolle's Theorem from these examples?

(i)  $f(x) = [x]$  for  $x \in [5, 9]$

(ii)  $f(x) = [x]$  for  $x \in [-2, 2]$

(iii)  $f(x) = x^2 - 1$  for  $x \in [1, 2]$

##### ANS:

$$(i) f(x) = [x] \text{ for } x \in [5, 9]$$

It is evident that the given function  $f(x)$  is not continuous at every integral point.

In particular,  $f(x)$  is not continuous at  $x = 5$  and  $x = 9$

$\Rightarrow f(x)$  is not continuous in  $[5, 9]$ .

$$\text{Also, } f(5) = [5] = 5 \text{ and } f(9) = [9] = 9$$

$$\therefore f(5) \neq f(9)$$

The differentiability of  $f$  in  $(5, 9)$  is checked as follows.

Let  $n$  be an integer such that  $n \in (5, 9)$ .

The left hand limit of  $f$  at  $x = n$  is,

$$\lim_{h \rightarrow 0^-} \frac{f(n+h) - f(n)}{h} = \lim_{h \rightarrow 0^-} \frac{[n+h] - [n]}{h} = \lim_{h \rightarrow 0^-} \frac{n-1-n}{h} = \lim_{h \rightarrow 0^-} \frac{-1}{h} = \infty$$

The right hand limit of  $f$  at  $x = n$  is,

$$\lim_{h \rightarrow 0^+} \frac{f(n+h) - f(n)}{h} = \lim_{h \rightarrow 0^+} \frac{[n+h] - [n]}{h} = \lim_{h \rightarrow 0^+} \frac{n-n}{h} = \lim_{h \rightarrow 0^+} 0 = 0$$

Since the left and right hand limits of  $f$  at  $x = n$  are not equal,  $f$  is not differentiable at  $x = n$

$\therefore f$  is not differentiable in  $(5, 9)$ .

It is observed that  $f$  does not satisfy all the conditions of the hypothesis of Rolle's Theorem.

Hence, Rolle's Theorem is not applicable for  $f(x) = [x]$  for  $x \in [5, 9]$ .

$$(ii) f(x) = [x] \text{ for } x \in [-2, 2]$$

It is evident that the given function  $f(x)$  is not continuous at every integral point.

In particular,  $f(x)$  is not continuous at  $x = -2$  and  $x = 2$

$\Rightarrow f(x)$  is not continuous in  $[-2, 2]$ .

$$\text{Also, } f(-2) = [-2] = -2 \text{ and } f(2) = [2] = 2$$

$$\therefore f(-2) \neq f(2)$$

The differentiability of  $f$  in  $(-2, 2)$  is checked as follows.

Let  $n$  be an integer such that  $n \in (-2, 2)$ .

The left hand limit of  $f$  at  $x = n$  is,

$$\lim_{h \rightarrow 0^-} \frac{f(n+h) - f(n)}{h} = \lim_{h \rightarrow 0^-} \frac{[n+h] - [n]}{h} = \lim_{h \rightarrow 0^-} \frac{n-1-n}{h} = \lim_{h \rightarrow 0^-} \frac{-1}{h} = \infty$$

The right hand limit of  $f$  at  $x = n$  is,

$$\lim_{h \rightarrow 0^+} \frac{f(n+h) - f(n)}{h} = \lim_{h \rightarrow 0^+} \frac{[n+h] - [n]}{h} = \lim_{h \rightarrow 0^+} \frac{n-n}{h} = \lim_{h \rightarrow 0^+} 0 = 0$$

Since the left and right hand limits of  $f$  at  $x = n$  are not equal,  $f$  is not differentiable at  $x = n$

$\therefore f$  is not differentiable in  $(-2, 2)$ .

It is observed that  $f$  does not satisfy all the conditions of the hypothesis of Rolle's Theorem.

Hence, Rolle's Theorem is not applicable for  $f(x) = [x]$  for  $x \in [-2, 2]$ .

(iii)  $f(x) = x^2 - 1$  for  $x \in [1, 2]$

It is evident that  $f$ , being a polynomial function, is continuous in  $[1, 2]$  and is differentiable in  $(1, 2)$ .

$$f(1) = (1)^2 - 1 = 0$$

$$f(2) = (2)^2 - 1 = 3$$

$\therefore f(1) \neq f(2)$

It is observed that  $f$  does not satisfy a condition of the hypothesis of Rolle's Theorem.

Hence, Rolle's Theorem is not applicable for  $f(x) = x^2 - 1$  for  $x \in [1, 2]$ .

**Question 3:**

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If  $f: [-5, 5] \rightarrow \mathbf{R}$  is a differentiable function and if  $f'(x)$  does not vanish anywhere, then prove that  $f(-5) \neq f(5)$ .

**ANS:**

It is given that  $f : [-5, 5] \rightarrow \mathbf{R}$  is a differentiable function.

Since every differentiable function is a continuous function, we obtain

(a)  $f$  is continuous on  $[-5, 5]$ .

(b)  $f$  is differentiable on  $(-5, 5)$ .

Therefore, by the Mean Value Theorem, there exists  $c \in (-5, 5)$  such that

$$f'(c) = \frac{f(5) - f(-5)}{5 - (-5)}$$

$$\Rightarrow 10f'(c) = f(5) - f(-5)$$

It is also given that  $f'(x)$  does not vanish anywhere.

$$\therefore f'(c) \neq 0$$

$$\Rightarrow 10f'(c) \neq 0$$

$$\Rightarrow f(5) - f(-5) \neq 0$$

$$\Rightarrow f(5) \neq f(-5)$$

Hence, proved.

#### Question 4:

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Verify Mean Value Theorem, if  $f(x) = x^2 - 4x - 3$  in the interval  $[a, b]$ , where  $a = 1$  and  $b = 4$ .

#### ANS:

The given function is  $f(x) = x^2 - 4x - 3$

$f$ , being a polynomial function, is continuous in  $[1, 4]$  and is differentiable in  $(1, 4)$  whose derivative is  $2x - 4$ .

$$f(1) = 1^2 - 4 \times 1 - 3 = -6, f(4) = 4^2 - 4 \times 4 - 3 = -3$$

$$\therefore \frac{f(b) - f(a)}{b - a} = \frac{f(4) - f(1)}{4 - 1} = \frac{-3 - (-6)}{3} = \frac{3}{3} = 1$$

Mean Value Theorem states that there is a point  $c \in (1, 4)$  such that  $f'(c) = 1$

$$f'(c) = 1$$

$$\Rightarrow 2c - 4 = 1$$

$$\Rightarrow c = \frac{5}{2}, \text{ where } c = \frac{5}{2} \in (1, 4)$$

Hence, Mean Value Theorem is verified for the given function.

**Question 5:**

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Verify Mean Value Theorem, if  $f(x) = x^3 - 5x^2 - 3x$  in the interval  $[a, b]$ , where  $a = 1$  and  $b = 3$ . Find all  $c \in (1, 3)$  for which  $f'(c) = 0$

**ANS:**

The given function  $f$  is  $f(x) = x^3 - 5x^2 - 3x$

$f$ , being a polynomial function, is continuous in  $[1, 3]$  and is differentiable in  $(1, 3)$  whose derivative is  $3x^2 - 10x - 3$ .

$$f(1) = 1^3 - 5 \times 1^2 - 3 \times 1 = -7, \quad f(3) = 3^3 - 5 \times 3^2 - 3 \times 3 = -27$$

$$\therefore \frac{f(b) - f(a)}{b - a} = \frac{f(3) - f(1)}{3 - 1} = \frac{-27 - (-7)}{3 - 1} = -10$$

Mean Value Theorem states that there exist a point  $c \in (1, 3)$  such that  $f'(c) = -10$

$$f'(c) = -10$$

$$\Rightarrow 3c^2 - 10c - 3 = 10$$

$$\Rightarrow 3c^2 - 10c + 7 = 0$$

$$\Rightarrow 3c^2 - 3c - 7c + 7 = 0$$

$$\Rightarrow 3c(c - 1) - 7(c - 1) = 0$$

$$\Rightarrow (c - 1)(3c - 7) = 0$$

$$\Rightarrow c = 1, \frac{7}{3}, \text{ where } c = \frac{7}{3} \in (1, 3)$$

Hence, Mean Value Theorem is verified for the given function and  $c = \frac{7}{3} \in (1, 3)$  is the only point for which  $f'(c) = 0$

**Question 6:**

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Examine the applicability of Mean Value Theorem for all three functions given in the above exercise 2.

**ANS:**

(i)  $f(x) = [x]$  for  $x \in [5, 9]$

It is evident that the given function  $f(x)$  is not continuous at every integral point.

In particular,  $f(x)$  is not continuous at  $x = 5$  and  $x = 9$

$\Rightarrow f(x)$  is not continuous in  $[5, 9]$ .

The differentiability of  $f$  in  $(5, 9)$  is checked as follows.

Let  $n$  be an integer such that  $n \in (5, 9)$ .

The left hand limit of  $f$  at  $x = n$  is,

$$\lim_{h \rightarrow 0^-} \frac{f(n+h) - f(n)}{h} = \lim_{h \rightarrow 0^-} \frac{[n+h] - [n]}{h} = \lim_{h \rightarrow 0^-} \frac{n-1-n}{h} = \lim_{h \rightarrow 0^-} \frac{-1}{h} = \infty$$

The right hand limit of  $f$  at  $x = n$  is,

$$\lim_{h \rightarrow 0^+} \frac{f(n+h) - f(n)}{h} = \lim_{h \rightarrow 0^+} \frac{[n+h] - [n]}{h} = \lim_{h \rightarrow 0^+} \frac{n-n}{h} = \lim_{h \rightarrow 0^+} 0 = 0$$

Since the left and right hand limits of  $f$  at  $x = n$  are not equal,  $f$  is not differentiable at  $x = n$

$\therefore f$  is not differentiable in  $(5, 9)$ .

It is observed that  $f$  does not satisfy all the conditions of the hypothesis of Mean Value Theorem.

Hence, Mean Value Theorem is not applicable for  $f(x) = [x]$  for  $x \in [5, 9]$ .

(ii)  $f(x) = [x]$  for  $x \in [-2, 2]$

It is evident that the given function  $f(x)$  is not continuous at every integral point.

In particular,  $f(x)$  is not continuous at  $x = -2$  and  $x = 2$

$\Rightarrow f(x)$  is not continuous in  $[-2, 2]$ .

The differentiability of  $f$  in  $(-2, 2)$  is checked as follows.

Let  $n$  be an integer such that  $n \in (-2, 2)$ .

The left hand limit of  $f$  at  $x = n$  is,

$$\lim_{h \rightarrow 0^-} \frac{f(n+h) - f(n)}{h} = \lim_{h \rightarrow 0^-} \frac{[n+h] - [n]}{h} = \lim_{h \rightarrow 0^-} \frac{n-1-n}{h} = \lim_{h \rightarrow 0^-} \frac{-1}{h} = \infty$$

The right hand limit of  $f$  at  $x = n$  is,

$$\lim_{h \rightarrow 0^+} \frac{f(n+h) - f(n)}{h} = \lim_{h \rightarrow 0^+} \frac{[n+h] - [n]}{h} = \lim_{h \rightarrow 0^+} \frac{n-n}{h} = \lim_{h \rightarrow 0^+} 0 = 0$$

Since the left and right hand limits of  $f$  at  $x = n$  are not equal,  $f$  is not differentiable at  $x = n$

$\therefore f$  is not differentiable in  $(-2, 2)$ .

It is observed that  $f$  does not satisfy all the conditions of the hypothesis of Mean Value Theorem.

Hence, Mean Value Theorem is not applicable for  $f(x) = [x]$  for  $x \in [-2, 2]$ .

(iii)  $f(x) = x^2 - 1$  for  $x \in [1, 2]$

It is evident that  $f$ , being a polynomial function, is continuous in  $[1, 2]$  and is differentiable in  $(1, 2)$ .

It is observed that  $f$  satisfies all the conditions of the hypothesis of Mean Value Theorem.

Hence, Mean Value Theorem is applicable for  $f(x) = x^2 - 1$  for  $x \in [1, 2]$ .

It can be proved as follows.

$$f(1) = 1^2 - 1 = 0, \quad f(2) = 2^2 - 1 = 3$$

$$\therefore \frac{f(b) - f(a)}{b - a} = \frac{f(2) - f(1)}{2 - 1} = \frac{3 - 0}{1} = 3$$

$$f'(x) = 2x$$

$$\therefore f'(c) = 3$$

$$\Rightarrow 2c = 3$$

$$\Rightarrow c = \frac{3}{2} = 1.5, \text{ where } 1.5 \in [1, 2]$$