CLASS XII CHAPTER 1 RELATION AND FUNCTION

NCERT EX 1.3 SOLUTIONS

1. Let $f: \{1, 3, 4\} \to \{1, 2, 5\}$ and $g: \{1, 2, 5\} \to \{1, 3\}$ be given by $f = \{(1, 2), (3, 5), (4, 1)\}$ and $g = \{(1, 3), (2, 3), (5, 1)\}$. Write down gof.

ANS:

The functions $f: \{1, 3, 4\} \to \{1, 2, 5\}$ and $g: \{1, 2, 5\} \to \{1, 3\}$ are defined as

$f = \{(1, 2), (3, 5), (4, 1)\}$ and $g = \{(1, 3), (2, 3), (5, 1)\}$.

$gof(1) = g(f(1)) = g(2) = 3 \qquad [f(1) = 2 \text{ and } g(2) = 3]$

$gof(3) = g(f(3)) = g(5) = 1 \qquad [f(3) = 5 \text{ and } g(5) = 1]$

$gof(4) = g(f(4)) = g(1) = 3 \qquad [f(4) = 1 \text{ and } g(1) = 3]$

$\therefore gof = \{(1,3),(3,1),(4,3)\}$

2. Let f, g and h be functions from **R** to **R**. Show that

$$(f + g)oh = foh + goh$$
$$(f \, . \, g)oh = (foh) \, . \, (goh)$$

ANS:

To prove:

$(f + g)oh = foh + goh$

Consider:

$((f+g)oh)(x)$

$= (f+g)(h(x))$

$= f(h(x)) + g(h(x))$

$= (foh)(x) + (goh)(x)$

$= \{(foh)+(goh)\}(x)$

$\therefore ((f+g)oh)(x) = \{(foh)+(goh)\}(x) \qquad \forall x \in \mathbf{R}$

Hence, $(f+g)oh = foh + goh$.

To prove:

$(f \cdot g)oh = (foh)\cdot(goh)$

Consider:

$((f \cdot g)oh)(x)$

$= (f \cdot g)(h(x))$

$= f(h(x)).g(h(x))$

$= (foh)(x).(goh)(x)$

$= \{(foh).(goh)\}(x)$

$\therefore ((f \cdot g)oh)(x) = \{(foh).(goh)\}(x) \; \forall x \in \mathrm{R}$

Hence, $(f \cdot g)oh = (foh)\cdot(goh)$.

3. Find gof and fog, if

(i) $f(x) = |x|$ and $g(x) = |5x - 2|$

(ii) $f(x) = 8x^3$ and $g(x) = x^{\frac{1}{3}}$.

AS:

(i) $f(x) = |x|$ and $g(x) = |5x-2|$

$$\therefore (gof)(x) = g(f(x)) = g(|x|) = |5|x|-2|$$
$$(fog)(x) = f(g(x)) = f(|5x-2|) = ||5x-2|| = |5x-2|$$

(ii) $f(x) = 8x^3$ and $g(x) = x^{\frac{1}{3}}$

$$\therefore (gof)(x) = g(f(x)) = g(8x^3) = (8x^3)^{\frac{1}{3}} = 2x$$
$$(fog)(x) = f(g(x)) = f\left(x^{\frac{1}{3}}\right) = 8\left(x^{\frac{1}{3}}\right)^3 = 8x$$

4. If $f(x) = \dfrac{(4x+3)}{(6x-4)}$, $x \neq \dfrac{2}{3}$, show that $fof(x) = x$, for all $x \neq \dfrac{2}{3}$. What is the inverse of f?

ANS:

It is given that $f(x) = \dfrac{(4x+3)}{(6x-4)}$, $x \neq \dfrac{2}{3}$.

$$(fof)(x) = f(f(x)) = f\left(\frac{4x+3}{6x-4}\right)$$
$$= \frac{4\left(\frac{4x+3}{6x-4}\right)+3}{6\left(\frac{4x+3}{6x-4}\right)-4} = \frac{16x+12+18x-12}{24x+18-24x+16} = \frac{34x}{34} = x$$

Therefore, $fof(x) = x$, for all $x \neq \dfrac{2}{3}$.

$\Rightarrow fof = I$

Hence, the given function f is invertible and the inverse of f is f itself.

5. **State with reason whether following functions have inverse**

(i) $f: \{1, 2, 3, 4\} \to \{10\}$ with

$f = \{(1, 10), (2, 10), (3, 10), (4, 10)\}$

(ii) $g: \{5, 6, 7, 8\} \to \{1, 2, 3, 4\}$ with

$g = \{(5, 4), (6, 3), (7, 4), (8, 2)\}$

(iii) $h: \{2, 3, 4, 5\} \to \{7, 9, 11, 13\}$ with

$h = \{(2, 7), (3, 9), (4, 11), (5, 13)\}$

ANS:

(i) $f: \{1, 2, 3, 4\} \to \{10\}$ defined as:

$f = \{(1, 10), (2, 10), (3, 10), (4, 10)\}$

From the given definition of f, we can see that f is a many one function as: $f(1) = f(2) = f(3) = f(4) = 10$

$\therefore f$ is not one-one.

Hence, function f does not have an inverse.

(ii) $g: \{5, 6, 7, 8\} \to \{1, 2, 3, 4\}$ defined as:

$g = \{(5, 4), (6, 3), (7, 4), (8, 2)\}$

From the given definition of g, it is seen that g is a many one function as: $g(5) = g(7) = 4$.

$\therefore g$ is not one-one,

Hence, function g does not have an inverse.

(iii) $h: \{2, 3, 4, 5\} \to \{7, 9, 11, 13\}$ defined as:

$h = \{(2, 7), (3, 9), (4, 11), (5, 13)\}$

It is seen that all distinct elements of the set $\{2, 3, 4, 5\}$ have distinct images under h.

$\therefore$Function h is one-one.

Also, h is onto since for every element y of the set $\{7, 9, 11, 13\}$, there exists an element x in the set $\{2, 3, 4, 5\}$ such that $h(x) = y$.

Thus, h is a one-one and onto function. Hence, h has an inverse.

6. Show that $f : [-1, 1] \to \mathbf{R}$, given by $f(x) = \dfrac{x}{(x+2)}$ is one-one. Find the inverse of the function $f : [-1, 1] \to$ Range f.

f: $[-1, 1] \to \mathrm{R}$ is given as $f(x) = \dfrac{x}{(x+2)}$.

Let $f(x) = f(y)$.

$$\Rightarrow \frac{x}{x+2} = \frac{y}{y+2}$$
$$\Rightarrow xy + 2x = xy + 2y$$
$$\Rightarrow 2x = 2y$$
$$\Rightarrow x = y$$

$\therefore f$ is a one-one function.

It is clear that f: $[-1, 1] \to$ Range f is onto.

$\therefore f$: $[-1, 1] \to$ Range f is one-one and onto and therefore, the inverse of the function:

f: $[-1, 1] \to$ Range f exists.

Let g: Range $f \to [-1, 1]$ be the inverse of f.

Let y be an arbitrary element of range f.

Since f: $[-1, 1] \to$ Range f is onto, we have:

$$y = f(x) \text{ for same } x \in [-1, 1]$$
$$\Rightarrow y = \frac{x}{x+2}$$
$$\Rightarrow xy + 2y = x$$
$$\Rightarrow x(1-y) = 2y$$
$$\Rightarrow x = \frac{2y}{1-y},\ y \neq 1$$

Now, let us define g: Range $f \to [-1, 1]$ as

$$g(y) = \frac{2y}{1-y},\ y \neq 1.$$

$$\text{Now, } (gof)(x) = g(f(x)) = g\left(\frac{x}{x+2}\right) = \frac{2\left(\frac{x}{x+2}\right)}{1-\frac{x}{x+2}} = \frac{2x}{x+2-x} = \frac{2x}{2} = x$$

$$(fog)(y) = f(g(y)) = f\left(\frac{2y}{1-y}\right) = \frac{\frac{2y}{1-y}}{\frac{2y}{1-y}+2} = \frac{2y}{2y+2-2y} = \frac{2y}{2} = y$$

$\therefore gof = \mathrm{I}_{[-1,\,1]}$ and $fog = \mathrm{I}_{\text{Range } f}$

$\therefore f^{-1} = g$

$$\Rightarrow f^{-1}(y) = \frac{2y}{1-y},\ y \neq 1$$

7. Consider $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = 4x + 3$. Show that f is invertible. Find the inverse of f.

ANS;

$f: \mathbf{R} \to \mathbf{R}$ is given by,

$f(x) = 4x + 3$

One-one:

Let $f(x) = f(y)$.

$\Rightarrow 4x+3 = 4y+3$
$\Rightarrow 4x = 4y$
$\Rightarrow x = y$

$\therefore f$ is a one-one function.

Onto:

For $y \in \mathbf{R}$, let $y = 4x + 3$.

$\Rightarrow x = \frac{y-3}{4} \in \mathbf{R}$

Therefore, for any $y \in \mathbf{R}$, there exists $x = \frac{y-3}{4} \in \mathbf{R}$ such that

$f(x) = f\left(\frac{y-3}{4}\right) = 4\left(\frac{y-3}{4}\right) + 3 = y.$

$\therefore f$ is onto.

Thus, f is one-one and onto and therefore, f^{-1} exists.

Let us define $g: \mathbf{R} \to \mathbf{R}$ by $g(x) = \frac{y-3}{4}$.

$$\text{Now, } (gof)(x) = g(f(x)) = g(4x+3) = \frac{(4x+3)-3}{4} = x$$

$$(fo\,g)(y) = f(g(y)) = f\left(\frac{y-3}{4}\right) = 4\left(\frac{y-3}{4}\right) + 3 = y-3+3 = y$$

$\therefore gof = fog = I_R$

Hence, f is invertible and the inverse of f is given by

$f^{-1}(y) = g(y) = \frac{y-3}{4}.$

8. Consider $f: \mathbf{R}_+ \to [4, \infty)$ given by $f(x) = x^2 + 4$. Show that f is invertible with the inverse f^{-1} of f given by $f^{-1}(y) = \sqrt{y-4}$, where $\mathbf{R}_+$ is the set of all non-negative real numbers.

ANS:

$f: \mathbf{R}_+ \to [4, \infty)$ is given as $f(x) = x^2 + 4$.

One-one:

Let $f(x) = f(y)$.

$\Rightarrow x^2 + 4 = y^2 + 4$

$\Rightarrow x^2 = y^2$

$\Rightarrow x = y \qquad [\text{as } x = y \in \mathbf{R}_+]$

$\therefore f$ is a one-one function.

Onto:

For $y \in [4, \infty)$, let $y = x^2 + 4$.

$\Rightarrow x^2 = y - 4 \geq 0 \qquad [\text{as } y \geq 4]$

$\Rightarrow x = \sqrt{y-4} \geq 0$

Therefore, for any $y \in \mathbf{R}$, there exists $x = \sqrt{y-4} \in \mathbf{R}$ such that

$f(x) = f\left(\sqrt{y-4}\right) = \left(\sqrt{y-4}\right)^2 + 4 = y - 4 + 4 = y$.

$\therefore f$ is onto.

Thus, f is one-one and onto and therefore, f^{-1} exists.

Let us define $g: [4, \infty) \to \mathbf{R}_+$ by,

$g(y) = \sqrt{y-4}$

Now, $gof(x) = g(f(x)) = g(x^2 + 4) = \sqrt{(x^2+4) - 4} = \sqrt{x^2} = x$

And, $fog(y) = f(g(y)) = f\left(\sqrt{y-4}\right) = \left(\sqrt{y-4}\right)^2 + 4 = (y-4) + 4 = y$

$\therefore gof = fog = I_{\mathbf{R}_+}$

Hence, f is invertible and the inverse of f is given by

$f^{-1}(y) = g(y) = \sqrt{y-4}$.

9. Consider $f: \mathbf{R}_+ \to [-5, \infty)$ given by $f(x) = 9x^2 + 6x - 5$. Show that f is invertible with $f^{-1}(y) = \left(\frac{(\sqrt{y+6})-1}{3}\right)$.

ANS:

f: $\mathbf{R}_+ \to [-5, \infty)$ is given as $f(x) = 9x^2 + 6x - 5$.

Let y be an arbitrary element of $[-5, \infty)$.

Let $y = 9x^2 + 6x - 5$.

$$\Rightarrow y = (3x+1)^2 - 1 - 5 = (3x+1)^2 - 6$$
$$\Rightarrow (3x+1)^2 = y+6$$
$$\Rightarrow 3x+1 = \sqrt{y+6} \qquad [\text{as } y \geq -5 \Rightarrow y+6 > 0]$$
$$\Rightarrow x = \frac{\sqrt{y+6}-1}{3}$$

$\therefore f$ is onto, thereby range $f = [-5, \infty)$.

Let us define g: $[-5, \infty) \to \mathbf{R}_+$ as $g(y) = \frac{\sqrt{y+6}-1}{3}$.

We now have:

$$\begin{aligned}(gof)(x) = g(f(x)) &= g(9x^2+6x-5) \\ &= g\left((3x+1)^2 - 6\right) \\ &= \frac{\sqrt{(3x+1)^2 - 6 + 6} - 1}{3} \\ &= \frac{3x+1-1}{3} = x\end{aligned}$$

$$\text{And, } (fog)(y) = f(g(y)) = f\left(\frac{\sqrt{y+6}-1}{3}\right)$$

$$\begin{aligned}&= \left[3\left(\frac{\sqrt{y+6}-1}{3}\right)+1\right]^2 - 6 \\ &= \left(\sqrt{y+6}\right)^2 - 6 = y+6-6 = y\end{aligned}$$

$\therefore gof = I_{\mathbf{R}_+}$ and $fo\,g = I_{[-5,\infty)}$

Hence, f is invertible and the inverse of f is given by

$$f^{-1}(y) = g(y) = \frac{\sqrt{y+6}-1}{3}.$$

10. Let $f : X \to Y$ be an invertible function. Show that f has unique inverse. (Hint: suppose g_1 and g_2 are two inverses of f. Then for all $y \in Y$, $fog_1(y) = 1_Y(y) = fog_2(y)$. Use one-one ness of f).

ANS:

Let f: $X \to Y$ be an invertible function.

Also, suppose f has two inverses (say g_1 and g_2).

Then, for all $y \in Y$, we have:

$$fog_1(y) = I_Y(y) = fog_2(y)$$
$$\Rightarrow f(g_1(y)) = f(g_2(y))$$
$$\Rightarrow g_1(y) = g_2(y) \qquad [f \text{ is invertible} \Rightarrow f \text{ is one-one}]$$
$$\Rightarrow g_1 = g_2 \qquad [g \text{ is one-one}]$$

Hence, f has a unique inverse.

11. Consider $f : \{1, 2, 3\} \to \{a, b, c\}$ given by $f(1) = a$, $f(2) = b$ and $f(3) = c$. Find f^{-1} and show that $(f^{-1})^{-1} = f$.

ANS:

Function f: $\{1, 2, 3\} \to \{a, b, c\}$ is given by,

$f(1) = a$, $f(2) = b$, and $f(3) = c$

If we define g: $\{a, b, c\} \to \{1, 2, 3\}$ as $g(a) = 1$, $g(b) = 2$, $g(c) = 3$, then we have:

$$(fog)(a) = f(g(a)) = f(1) = a$$
$$(fog)(b) = f(g(b)) = f(2) = b$$
$$(fog)(c) = f(g(c)) = f(3) = c$$

And,

$$(gof)(1) = g(f(1)) = g(a) = 1$$
$$(gof)(2) = g(f(2)) = g(b) = 2$$
$$(gof)(3) = g(f(3)) = g(c) = 3$$

$\therefore gof = I_X$ and $fog = I_Y$, where $X = \{1, 2, 3\}$ and $Y = \{a, b, c\}$.

Thus, the inverse of f exists and $f^{-1} = g$.

$\therefore f^{-1}$: $\{a, b, c\} \to \{1, 2, 3\}$ is given by,

$f^{-1}(a) = 1$, $f^{-1}(b) = 2$, $f^{-1}(c) = 3$

Let us now find the inverse of f^{-1} i.e., find the inverse of g.

If we define h: $\{1, 2, 3\} \to \{a, b, c\}$ as

$h(1) = a$, $h(2) = b$, $h(3) = c$, then we have:

$(goh)(1) = g(h(1)) = g(a) = 1$
$(goh)(2) = g(h(2)) = g(b) = 2$
$(goh)(3) = g(h(3)) = g(c) = 3$

And,

$(hog)(a) = h(g(a)) = h(1) = a$
$(hog)(b) = h(g(b)) = h(2) = b$
$(hog)(c) = h(g(c)) = h(3) = c$

$\therefore goh = I_X$ and $hog = I_Y$, where $X = \{1, 2, 3\}$ and $Y = \{a, b, c\}$.

Thus, the inverse of g exists and $g^{-1} = h \Rightarrow (f^{-1})^{-1} = h$.

It can be noted that $h = f$.

Hence, $(f^{-1})^{-1} = f$.

12. Let f: X → Y be an invertible function. Show that the inverse of f^{-1} is f, i.e., $(f^{-1})^{-1} = f$.

ANS:

Let f: $X \rightarrow Y$ be an invertible function.

Then, there exists a function g: $Y \rightarrow X$ such that $gof = I_X$ and $fog = I_Y$.

Here, $f^{-1} = g$.

Now, $gof = I_X$ and $fog = I_Y$

$\Rightarrow f^{-1}of = I_X$ and $fof^{-1} = I_Y$

Hence, f^{-1}: $Y \rightarrow X$ is invertible and f is the inverse of f^{-1}

i.e., $(f^{-1})^{-1} = f$.

13. If $f: \mathbf{R} \to \mathbf{R}$ be given by $f(x) = (3 - x^3)^{\frac{1}{3}}$, then $fof(x)$ is

(A) $x^{\frac{1}{3}}$ (B) x^3 (C) x (D) $(3 - x^3)$.

ANS:

$f: \mathbf{R} \to \mathbf{R}$ is given as $f(x) = (3 - x^3)^{\frac{1}{3}}$.

$$f(x) = (3 - x^3)^{\frac{1}{3}}$$

$$\therefore fof(x) = f(f(x)) = f\left((3 - x^3)^{\frac{1}{3}}\right) = \left[3 - \left((3 - x^3)^{\frac{1}{3}}\right)^3\right]^{\frac{1}{3}}$$

$$= \left[3 - (3 - x^3)\right]^{\frac{1}{3}} = (x^3)^{\frac{1}{3}} = x$$

$$\therefore fof(x) = x$$

The correct answer is C.

14. Let $f: \mathbf{R} - \left\{-\frac{4}{3}\right\} \to \mathbf{R}$ be a function defined as $f(x) = \frac{4x}{3x+4}$. The inverse of f is the map g : Range $f \to \mathbf{R} - \left\{-\frac{4}{3}\right\}$ given by

(A) $g(y) = \frac{3y}{3-4y}$ (B) $g(y) = \frac{4y}{4-3y}$

(C) $g(y) = \frac{4y}{3-4y}$ (D) $g(y) = \frac{3y}{4-3y}$

ANS:

It is given that $f : \mathbf{R} - \left\{-\frac{4}{3}\right\} \to \mathbf{R}$ is defined as $f(x) = \frac{4x}{3x+4}$.

Let y be an arbitrary element of Range f.

Then, there exists $x \in \mathbf{R} - \left\{-\frac{4}{3}\right\}$ such that $y = f(x)$.

$$\Rightarrow y = \frac{4x}{3x+4}$$

$$\Rightarrow 3xy + 4y = 4x$$

$$\Rightarrow x(4 - 3y) = 4y$$

$$\Rightarrow x = \frac{4y}{4-3y}$$

Let us define g: Range $f \to \mathbf{R} - \left\{-\frac{4}{3}\right\}$ as $g(y) = \frac{4y}{4-3y}$.

Now, $(g \circ f)(x) = g(f(x)) = g\left(\frac{4x}{3x+4}\right)$

$$= \frac{4\left(\frac{4x}{3x+4}\right)}{4-3\left(\frac{4x}{3x+4}\right)} = \frac{16x}{12x+16-12x} = \frac{16x}{16} = x$$

$$\text{And,} (f \circ g)(y) = f(g(y)) = f\left(\frac{4y}{4-3y}\right)$$

$$= \frac{4\left(\frac{4y}{4-3y}\right)}{3\left(\frac{4y}{4-3y}\right)+4} = \frac{16y}{12y+16-12y} = \frac{16y}{16} = y$$

$\therefore gof = I_{\mathbf{R}-\left\{-\frac{4}{3}\right\}}$ and $fog = I_{\text{Range } f}$

Thus, g is the inverse of f i.e., $f^{-1} = g$.

Hence, the inverse of f is the map g: Range $f \to \mathbf{R} - \left\{-\frac{4}{3}\right\}$, which is given by

$$g(y) = \frac{4y}{4-3y}.$$

The correct answer is B.