

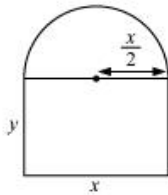
## SOLUTIONS TO CLASS TEST XII C

### WORD PROBLEMS (MAX/MIN.) HELD ON 23 AUG 2011

ANS 1.

Let  $x$  and  $y$  be the length and breadth of the rectangular window.

Radius of the semicircular opening =  $\frac{x}{2}$



It is given that the perimeter of the window is 10 m.

$$\therefore x + 2y + \frac{\pi x}{2} = 10$$

$$\Rightarrow x \left( 1 + \frac{\pi}{2} \right) + 2y = 10$$

$$\Rightarrow 2y = 10 - x \left( 1 + \frac{\pi}{2} \right)$$

$$\Rightarrow y = 5 - x \left( \frac{1}{2} + \frac{\pi}{4} \right)$$

$\therefore$  Area of the window ( $A$ ) is given by,

$$A = xy + \frac{\pi}{2} \left( \frac{x}{2} \right)^2$$

$$= x \left[ 5 - x \left( \frac{1}{2} + \frac{\pi}{4} \right) \right] + \frac{\pi}{8} x^2$$

$$= 5x - x^2 \left( \frac{1}{2} + \frac{\pi}{4} \right) + \frac{\pi}{8} x^2$$

$$\therefore \frac{dA}{dx} = 5 - 2x \left( \frac{1}{2} + \frac{\pi}{4} \right) + \frac{\pi}{4} x$$

$$= 5 - x \left( 1 + \frac{\pi}{2} \right) + \frac{\pi}{4} x$$

$$\therefore \frac{d^2A}{dx^2} = - \left( 1 + \frac{\pi}{2} \right) + \frac{\pi}{4} = -1 - \frac{\pi}{4}$$

$$\begin{aligned}
\text{Now, } \frac{dA}{dx} &= 0 \\
\Rightarrow 5 - x\left(1 + \frac{\pi}{2}\right) + \frac{\pi}{4}x &= 0 \\
\Rightarrow 5 - x - \frac{\pi}{4}x &= 0 \\
\Rightarrow x\left(1 + \frac{\pi}{4}\right) &= 5 \\
\Rightarrow x = \frac{5}{\left(1 + \frac{\pi}{4}\right)} &= \frac{20}{\pi + 4}
\end{aligned}$$

Thus, when  $x = \frac{20}{\pi + 4}$  then  $\frac{d^2A}{dx^2} < 0$ .

Therefore, by second derivative test, the area is the maximum when length  $x = \frac{20}{\pi + 4}$  m.

Now,

$$y = 5 - \frac{20}{\pi + 4} \left(\frac{2 + \pi}{4}\right) = 5 - \frac{5(2 + \pi)}{\pi + 4} = \frac{10}{\pi + 4} \text{ m}$$

Hence, the required dimensions of the window to admit maximum light is given by length  $= \frac{20}{\pi + 4}$  m and breadth  $= \frac{10}{\pi + 4}$  m.

## ANS 2.

Let a piece of length  $l$  be cut from the given wire to make a square.

Then, the other piece of wire to be made into a circle is of length  $(28 - l)$  m.

Now, side of square  $= \frac{l}{4}$ .

Let  $r$  be the radius of the circle. Then,  $2\pi r = 28 - l \Rightarrow r = \frac{1}{2\pi}(28 - l)$ .

The combined areas of the square and the circle ( $A$ ) is given by,

$$\begin{aligned}
A &= (\text{side of the square})^2 + r^2 \\
&= \frac{l^2}{16} + \pi \left[ \frac{1}{2\pi}(28 - l) \right]^2 \\
&= \frac{l^2}{16} + \frac{1}{4\pi}(28 - l)^2 \\
\therefore \frac{dA}{dl} &= \frac{2l}{16} + \frac{2}{4\pi}(28 - l)(-1) = \frac{l}{8} - \frac{1}{2\pi}(28 - l) \\
\frac{d^2A}{dl^2} &= \frac{1}{8} + \frac{1}{2\pi} > 0 \\
\text{Now, } \frac{dA}{dl} = 0 &\Rightarrow \frac{l}{8} - \frac{1}{2\pi}(28 - l) = 0 \\
\Rightarrow \frac{\pi l - 4(28 - l)}{8\pi} &= 0 \\
\Rightarrow (\pi + 4)l - 112 &= 0 \\
\Rightarrow l &= \frac{112}{\pi + 4}
\end{aligned}$$

Thus, when  $l = \frac{112}{\pi+4}$ ,  $\frac{d^2A}{dl^2} > 0$ .

∴ By second derivative test, the area ( $A$ ) is the minimum when  $l = \frac{112}{\pi+4}$ .

Hence, the combined area is the minimum when the length of the wire in making the square is  $\frac{112}{\pi+4}$  cm while the length of the wire in making the circle is  $28 - \frac{112}{\pi+4} = \frac{28\pi}{\pi+4}$  cm.

**ANS3.**

Let  $r$  and  $h$  be the radius and the height (altitude) of the cone respectively.

Then, the volume ( $V$ ) of the cone is given as:

$$V = \frac{1}{3}\pi r^2 h \Rightarrow h = \frac{3V}{r^2}$$

The surface area ( $S$ ) of the cone is given by,

$$S = \pi r l \text{ (where } l \text{ is the slant height)}$$

$$\begin{aligned} &= \pi r \sqrt{r^2 + h^2} \\ &= \pi r \sqrt{r^2 + \frac{9V^2}{r^4}} = \frac{\pi r \sqrt{9r^6 + V^2}}{r^2} \\ &= \frac{1}{r} \sqrt{\pi^2 r^6 + 9V^2} \end{aligned}$$

$$\begin{aligned} \therefore \frac{dS}{dr} &= \frac{r \cdot \frac{6\pi^2 r^5}{2\sqrt{\pi^2 r^6 + 9V^2}} - \sqrt{\pi^2 r^6 + 9V^2}}{r^2} \\ &= \frac{3\pi^2 r^6 - \pi^2 r^6 - 9V^2}{r^2 \sqrt{\pi^2 r^6 + 9V^2}} \\ &= \frac{2\pi^2 r^6 - 9V^2}{r^2 \sqrt{\pi^2 r^6 + 9V^2}} \\ &= \frac{2\pi^2 r^6 - 9V^2}{r^2 \sqrt{\pi^2 r^6 + 9V^2}} \end{aligned}$$

$$\text{Now, } \frac{dS}{dr} = 0 \Rightarrow 2\pi^2 r^6 = 9V^2 \Rightarrow r^6 = \frac{9V^2}{2\pi^2}$$

Thus, it can be easily verified that when  $r^6 = \frac{9V^2}{2\pi^2}$ ,  $\frac{d^2S}{dr^2} > 0$ .

∴ By second derivative test, the surface area of the cone is the least when  $r^6 = \frac{9V^2}{2\pi^2}$ .

$$\text{When } r^6 = \frac{9V^2}{2\pi^2}, h = \frac{3V}{\pi r^2} = \frac{3}{\pi r^2} \left( \frac{2\pi^2 r^6}{9} \right)^{\frac{1}{2}} = \frac{3}{\pi r^2} \cdot \frac{\sqrt{2}\pi r^3}{3} = \sqrt{2}r.$$

Hence, for a given volume, the right circular cone of the least curved surface has an altitude equal to  $\sqrt{2}$  times the radius of the base.

**ANS 4. TRY YOURSELF**