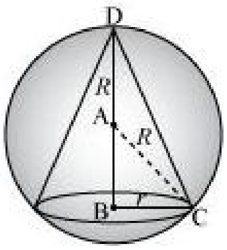
SOLUTIONS TO CLASS TEST XII B

WORD PROBLEMS (MAX/MIN.) HELD ON 23 AUG 2011

ANS1.

Let r and h be the radius and height of the cone respectively inscribed in a sphere of radius R.



Let V be the volume of the cone.

Then, $V = \frac{1}{3}\pi r^2 h$

Height of the cone is given by,

$h = R + AB = R + \sqrt{R^2 - r^2}$ [ABC is a right triangle]

$$\therefore V = \frac{1}{3}\pi r^2\left(R + \sqrt{R^2 - r^2}\right)$$

$$= \frac{1}{3}\pi r^2 R + \frac{1}{3}\pi r^2\sqrt{R^2 - r^2}$$

$$\therefore \frac{dV}{dr} = \frac{2}{3}\pi r R + \frac{2}{3}\pi r\sqrt{R^2 - r^2} + \frac{1}{3}\pi r^2 \cdot \frac{(-2r)}{2\sqrt{R^2 - r^2}}$$

$$= \frac{2}{3}\pi r R + \frac{2}{3}\pi r\sqrt{R^2 - r^2} - \frac{1}{3}\pi\frac{r^3}{\sqrt{R^2 - r^2}}$$

$$= \frac{2}{3}\pi r R + \frac{2\pi r\left(R^2 - r^2\right) - \pi r^3}{3\sqrt{R^2 - r^2}}$$

$$= \frac{2}{3}\pi r R + \frac{2\pi r R^2 - 3\pi r^3}{3\sqrt{R^2 - r^2}}$$

$$\frac{d^2V}{dr^2}=\frac{2\pi R}{3}+\frac{3\sqrt{R^2-r^2}\left(2\pi R^2-9\pi r^2\right)-\left(2\pi rR^2-3\pi r^3\right)\cdot\frac{(-2r)}{6\sqrt{R^2-r^2}}}{9\left(R^2-r^2\right)}$$

$$=\frac{2}{3}\pi R+\frac{9\left(R^2-r^2\right)\left(2\pi R^2-9\pi r^2\right)+2\pi r^2R^2+3\pi r^4}{27\left(R^2-r^2\right)^{\frac{3}{2}}}$$

Now, $\frac{dV}{dr}=0 \Rightarrow \frac{2}{3}\pi\ rR=\frac{3\pi r^3-2\pi rR^2}{3\sqrt{R^2-r^2}}$

$\Rightarrow 2R=\frac{3r^2-2R^2}{\sqrt{R^2-r^2}} \Rightarrow 2R\sqrt{R^2-r^2}=3r^2-2R^2$

$\Rightarrow 4R^2\left(R^2-r^2\right)=\left(3r^2-2R^2\right)^2$

$\Rightarrow 4R^4-4R^2r^2=9r^4+4R^4-12r^2R^2$

$\Rightarrow 9r^4=8R^2r^2$

$\Rightarrow r^2=\frac{8}{9}R^2$

When $r^2=\frac{8}{9}R^2$, then $\frac{d^2V}{dr^2}<0$.

$\therefore$ By second derivative test, the volume of the cone is the maximum when $r^2=\frac{8}{9}R^2$.

When $r^2=\frac{8}{9}R^2$, $h=R+\sqrt{R^2-\frac{8}{9}R^2}=R+\sqrt{\frac{1}{9}R^2}=R+\frac{R}{3}=\frac{4}{3}R$.

Therefore,

$=\frac{1}{3}\pi\left(\frac{8}{9}R^2\right)\left(\frac{4}{3}R\right)$

$=\frac{8}{27}\left(\frac{4}{3}\pi R^3\right)$

$=\frac{8}{27}\times\left(\text{Volume of the sphere}\right)$

Hence, the volume of the largest cone that can be inscribed in the sphere is $\frac{8}{27}$ the volume of the sphere.

ANS 2.

Let r and h be the radius and height of the cylinder respectively.

Then, volume (V) of the cylinder is given by,

$$V = \pi r^2 h = 100 \qquad \text{(given)}$$

$$\therefore h = \frac{100}{\pi r^2}$$

Surface area (S) of the cylinder is given by,

$$S = 2\pi r^2 + 2\pi r h = 2\pi r^2 + \frac{200}{r}$$

$$\therefore \frac{dS}{dr} = 4\pi r - \frac{200}{r^2}, \quad \frac{d^2S}{dr^2} = 4\pi + \frac{400}{r^3}$$

$$\frac{dS}{dr} = 0 \Rightarrow 4\pi r = \frac{200}{r^2}$$

$$\Rightarrow r^3 = \frac{200}{4\pi} = \frac{50}{\pi}$$

$$\Rightarrow r = \left(\frac{50}{\pi}\right)^{\frac{1}{3}}$$

Now, it is observed that when $r = \left(\frac{50}{\pi}\right)^{\frac{1}{3}}$, $\frac{d^2S}{dr^2} > 0.$

$\therefore$By second derivative test, the surface area is the minimum when the radius of the cylinder is $\left(\frac{50}{\pi}\right)^{\frac{1}{3}}$ cm.

When $r = \left(\frac{50}{\pi}\right)^{\frac{1}{3}}$, $h = \dfrac{100}{\pi\left(\frac{50}{\pi}\right)^{\frac{2}{3}}} = \dfrac{2\times 50}{(50)^{\frac{2}{3}}(\pi)^{1-\frac{2}{3}}} = 2\left(\frac{50}{\pi}\right)^{\frac{1}{3}}$ cm.

Hence, the required dimensions of the can which has the minimum surface area is given by radius $= \left(\frac{50}{\pi}\right)^{\frac{1}{3}}$ cm and height $= 2\left(\frac{50}{\pi}\right)^{\frac{1}{3}}$ cm.

ANS 3.

Let r be the radius of the circle and a be the side of the square.

Then, we have:

$$2\pi r + 4a = k \text{ (where } k \text{ is constant)}$$
$$\Rightarrow a = \frac{k - 2\pi r}{4}$$

The sum of the areas of the circle and the square (A) is given by,

$$A = \pi r^2 + a^2 = \pi r^2 + \frac{(k - 2\pi r)^2}{16}$$
$$\therefore \frac{dA}{dr} = 2\pi r + \frac{2(k - 2\pi r)(-2\pi)}{16} = 2\pi r - \frac{\pi(k - 2\pi r)}{4}$$

$$\text{Now, } \frac{dA}{dr} = 0$$
$$\Rightarrow 2\pi r = \frac{\pi(k - 2\pi r)}{4}$$
$$8r = k - 2\pi r$$
$$\Rightarrow (8 + 2\pi) r = k$$
$$\Rightarrow r = \frac{k}{8 + 2\pi} = \frac{k}{2(4 + \pi)}$$
$$\text{Now, } \frac{d^2A}{dr^2} = 2\pi + \frac{\pi^2}{2} > 0$$
$$\therefore \text{ When } r = \frac{k}{2(4+\pi)},\ \frac{d^2A}{dr^2} > 0.$$

$\therefore$ The sum of the areas is least when $r = \frac{k}{2(4+\pi)}$.

When $r = \frac{k}{2(4+\pi)}$, $a = \frac{k - 2\pi\left[\frac{k}{2(4+\pi)}\right]}{4} = \frac{k(4+\pi) - \pi k}{4(4+\pi)4} = \frac{4k}{4(4+\pi)4} = \frac{k}{4+\pi} = 2r.$

Hence, it has been proved that the sum of their areas is least when the side of the square is double the radius of the circle.

$$A = xy + \frac{\pi}{2}\left(\frac{x}{2}\right)^2$$

$$= x\left[5 - x\left(\frac{1}{2}+\frac{\pi}{4}\right)\right]+\frac{\pi}{8}x^2$$

$$= 5x - x^2\left(\frac{1}{2}+\frac{\pi}{4}\right)+\frac{\pi}{8}x^2$$

$$\therefore \frac{dA}{dx} = 5 - 2x\left(\frac{1}{2}+\frac{\pi}{4}\right)+\frac{\pi}{4}x$$

$$= 5 - x\left(1+\frac{\pi}{2}\right)+\frac{\pi}{4}x$$

$$\therefore \frac{d^2A}{dx^2} = -\left(1+\frac{\pi}{2}\right)+\frac{\pi}{4} = -1-\frac{\pi}{4}$$

Now, $\frac{dA}{dx} = 0$

$$\Rightarrow 5 - x\left(1+\frac{\pi}{2}\right)+\frac{\pi}{4}x = 0$$

$$\Rightarrow 5 - x - \frac{\pi}{4}x = 0$$

$$\Rightarrow x\left(1+\frac{\pi}{4}\right) = 5$$

$$\Rightarrow x = \frac{5}{\left(1+\frac{\pi}{4}\right)} = \frac{20}{\pi+4}$$

Thus, when $x = \frac{20}{\pi+4}$ then $\frac{d^2A}{dx^2} < 0$.

Therefore, by second derivative test, the area is the maximum when length $x = \frac{20}{\pi+4}$ m.

Now,

$$y = 5 - \frac{20}{\pi+4}\left(\frac{2+\pi}{4}\right) = 5 - \frac{5(2+\pi)}{\pi+4} = \frac{10}{\pi+4} \text{ m}$$

Hence, the required dimensions of the window to admit maximum light is given by length $= \frac{20}{\pi+4}$ m and breadth $= \frac{10}{\pi+4}$ m.

ANS 4. TRY YOURSELF