

SOLUTIONS TO CLASS TEST XII A

WORD PROBLEMS (MAX/MIN.) HELD ON 23 AUG 2011

ANS 1.

Let r and h be the radius and height of the cylinder respectively.

Then, the surface area (S) of the cylinder is given by,

$$\begin{aligned} S &= 2\pi r^2 + 2\pi r h \\ \Rightarrow h &= \frac{S - 2\pi r^2}{2\pi r} \\ &= \frac{S}{2\pi} \left(\frac{1}{r} \right) - r \end{aligned}$$

Let V be the volume of the cylinder. Then,

$$V = \pi r^2 h = \pi r^2 \left[\frac{S}{2\pi} \left(\frac{1}{r} \right) - r \right] = \frac{S r}{2} - \pi r^3$$

$$\text{Then, } \frac{dV}{dr} = \frac{S}{2} - 3\pi r^2, \quad \frac{d^2V}{dr^2} = -6\pi r$$

$$\text{Now, } \frac{dV}{dr} = 0 \Rightarrow \frac{S}{2} = 3\pi r^2 \Rightarrow r^2 = \frac{S}{6\pi}$$

$$\text{When } r^2 = \frac{S}{6\pi}, \text{ then } \frac{d^2V}{dr^2} = -6\pi \left(\sqrt{\frac{S}{6\pi}} \right) < 0.$$

\therefore By second derivative test, the volume is the maximum when $r^2 = \frac{S}{6\pi}$.

$$\text{Now, when } r^2 = \frac{S}{6\pi}, \text{ then } h = \frac{6\pi r^2}{2\pi} \left(\frac{1}{r} \right) - r = 3r - r = 2r.$$

Hence, the volume is the maximum when the height is twice the radius i.e., when the height is equal to the diameter.

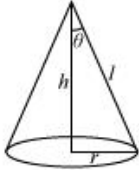
ANS 2.

Let θ be the semi-vertical angle of the cone.

It is clear that $\theta \in \left[0, \frac{\pi}{2}\right]$.

Let r , h , and l be the radius, height, and the slant height of the cone respectively.

The slant height of the cone is given as constant.



Now, $r = l \sin \theta$ and $h = l \cos \theta$

The volume (V) of the cone is given by,

$$\begin{aligned} V &= \frac{1}{3} \pi r^2 h \\ &= \frac{1}{3} \pi (l^2 \sin^2 \theta) (l \cos \theta) \\ &= \frac{1}{3} \pi l^3 \sin^2 \theta \cos \theta \\ \therefore \frac{dV}{d\theta} &= \frac{l^3 \pi}{3} [\sin^2 \theta (-\sin \theta) + \cos \theta (2 \sin \theta \cos \theta)] \\ &= \frac{l^3 \pi}{3} [-\sin^3 \theta + 2 \sin \theta \cos^2 \theta] \\ \frac{d^2V}{d\theta^2} &= \frac{l^3 \pi}{3} [-3 \sin^2 \theta \cos \theta + 2 \cos^3 \theta - 4 \sin^2 \theta \cos \theta] \\ &= \frac{l^3 \pi}{3} [2 \cos^3 \theta - 7 \sin^2 \theta \cos \theta] \end{aligned}$$

$$\text{Now, } \frac{dV}{d\theta} = 0$$

$$\Rightarrow \sin^3 \theta = 2 \sin \theta \cos^2 \theta$$

$$\Rightarrow \tan^2 \theta = 2$$

$$\Rightarrow \tan \theta = \sqrt{2}$$

$$\Rightarrow \theta = \tan^{-1} \sqrt{2}$$

Now, when $\theta = \tan^{-1} \sqrt{2}$, then $\tan^2 \theta = 2$ or $\sin^2 \theta = 2 \cos^2 \theta$.

Then, we have:

$$\frac{d^2V}{d\theta^2} = \frac{l^3 \pi}{3} [2 \cos^3 \theta - 14 \cos^3 \theta] = -4\pi l^3 \cos^3 \theta < 0 \text{ for } \theta \in \left[0, \frac{\pi}{2}\right]$$

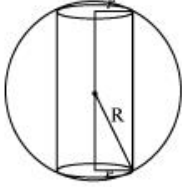
\therefore By second derivative test, the volume (V) is the maximum when $\theta = \tan^{-1} \sqrt{2}$.

Hence, for a given slant height, the semi-vertical angle of the cone of the maximum volume is $\tan^{-1} \sqrt{2}$.

ANS 3.

A sphere of fixed radius (R) is given.

Let r and h be the radius and the height of the cylinder respectively.



From the given figure, we have $h = 2\sqrt{R^2 - r^2}$.

The volume (V) of the cylinder is given by,

$$\begin{aligned} V &= \pi r^2 h = 2\pi r^2 \sqrt{R^2 - r^2} \\ \therefore \frac{dV}{dr} &= 4\pi r \sqrt{R^2 - r^2} + \frac{2\pi r^2 (-2r)}{2\sqrt{R^2 - r^2}} \\ &= 4\pi r \sqrt{R^2 - r^2} - \frac{2\pi r^3}{\sqrt{R^2 - r^2}} \\ &= \frac{4\pi r (R^2 - r^2) - 2\pi r^3}{\sqrt{R^2 - r^2}} \\ &= \frac{4\pi r R^2 - 6\pi r^3}{\sqrt{R^2 - r^2}} \end{aligned}$$

$$\text{Now, } \frac{dV}{dr} = 0 \Rightarrow 4\pi r R^2 - 6\pi r^3 = 0$$

$$\Rightarrow r^2 = \frac{2R^2}{3}$$

$$\begin{aligned} \text{Now, } \frac{d^2V}{dr^2} &= \frac{\sqrt{R^2 - r^2} (4\pi R^2 - 18\pi r^2) - (4\pi r R^2 - 6\pi r^3) \frac{(-2r)}{2\sqrt{R^2 - r^2}}}{(R^2 - r^2)} \\ &= \frac{(R^2 - r^2)(4\pi R^2 - 18\pi r^2) + r(4\pi r R^2 - 6\pi r^3)}{(R^2 - r^2)^{\frac{3}{2}}} \\ &= \frac{4\pi R^4 - 22\pi r^2 R^2 + 12\pi r^4 + 4\pi r^2 R^2}{(R^2 - r^2)^{\frac{3}{2}}} \end{aligned}$$

Now, it can be observed that at $r^2 = \frac{2R^2}{3}$, $\frac{d^2V}{dr^2} < 0$.

\therefore The volume is the maximum when $r^2 = \frac{2R^2}{3}$.

When $r^2 = \frac{2R^2}{3}$, the height of the cylinder is $2\sqrt{R^2 - \frac{2R^2}{3}} = 2\sqrt{\frac{R^2}{3}} = \frac{2R}{\sqrt{3}}$.

Hence, the volume of the cylinder is the maximum when the height of the cylinder is $\frac{2R}{\sqrt{3}}$.

ANS 4. TRY YOURSELF