CLASS XII APPLICATION OF DERIVATIVES CHAPTER 6

MISC. EX. SOLUTIONS

1. Using differentials, find the approximate value of each of the following:

(a) $\left(\frac{17}{81}\right)^{\frac{1}{4}}$ (b) $(33)^{-\frac{1}{5}}$

ANS :

(a) Consider $y = x^{\frac{1}{4}}$. Let $x = \frac{16}{81}$ and $\Delta x = \frac{1}{81}$.

Then, $\Delta y = (x + \Delta x)^{\frac{1}{4}} - x^{\frac{1}{4}}$

$$= \left(\frac{17}{81}\right)^{\frac{1}{4}} - \left(\frac{16}{81}\right)^{\frac{1}{4}}$$

$$= \left(\frac{17}{81}\right)^{\frac{1}{4}} - \frac{2}{3}$$

$$\therefore \left(\frac{17}{81}\right)^{\frac{1}{4}} = \frac{2}{3} + \Delta y$$

Now, dy is approximately equal to Δy and is given by,

$$dy = \left(\frac{dy}{dx}\right)\Delta x = \frac{1}{4(x)^{\frac{3}{4}}}(\Delta x) \qquad \left(\text{as } y = x^{\frac{1}{4}}\right)$$

$$= \frac{1}{4\left(\frac{16}{81}\right)^{\frac{3}{4}}}\left(\frac{1}{81}\right) = \frac{27}{4 \times 8} \times \frac{1}{81} = \frac{1}{32 \times 3} = \frac{1}{96} = 0.010$$

Hence, the approximate value of $\left(\frac{17}{81}\right)^{\frac{1}{4}}$ is $\frac{2}{3} + 0.010 = 0.667 + 0.010$

$= 0.677.$

(b) Consider $y = x^{-\frac{1}{5}}$. Let $x = 32$ and $\Delta x = 1$.

Then, $\Delta y = (x+\Delta x)^{-\frac{1}{5}} - x^{-\frac{1}{5}} = (33)^{-\frac{1}{5}} - (32)^{-\frac{1}{5}} = (33)^{-\frac{1}{5}} - \frac{1}{2}$

$\therefore (33)^{-\frac{1}{5}} = \frac{1}{2} + \Delta y$

Now, dy is approximately equal to Δy and is given by,

$$dy = \left(\frac{dy}{dx}\right)(\Delta x) = \frac{-1}{5(x)^{\frac{6}{5}}}(\Delta x) \qquad \left(\text{as } y = x^{-\frac{1}{5}}\right)$$

$$= -\frac{1}{5(2)^6}(1) = -\frac{1}{320} = -0.003$$

Hence, the approximate value of $(33)^{-\frac{1}{5}}$ is $\frac{1}{2} + (-0.003)$

$= 0.5 - 0.003 = 0.497$.

2. Show that the function given by $f(x) = \frac{\log x}{x}$ has maximum at $x = e$.

ANS :

The given function is $f(x) = \frac{\log x}{x}$.

$$f'(x) = \frac{x\left(\frac{1}{x}\right) - \log x}{x^2} = \frac{1-\log x}{x^2}$$

Now, $f'(x) = 0$

$\Rightarrow 1 - \log x = 0$

$\Rightarrow \log x = 1$

$\Rightarrow \log x = \log e$

$\Rightarrow x = e$

$$\text{Now, } f''(x) = \frac{x^2\left(-\frac{1}{x}\right) - (1-\log x)(2x)}{x^4}$$

$$= \frac{-x - 2x(1-\log x)}{x^4}$$

$$= \frac{-3 + 2\log x}{x^3}$$

$$\text{Now, } f''(e) = \frac{-3 + 2\log e}{e^3} = \frac{-3+2}{e^3} = \frac{-1}{e^3} < 0$$

Therefore, by second derivative test, f is the maximum at $x = e$.

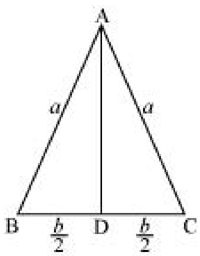
3. The two equal sides of an isosceles triangle with fixed base b are decreasing at the rate of 3 cm per second. How fast is the area decreasing when the two equal sides are equal to the base ?

ANS :

Let ΔABC be isosceles where BC is the base of fixed length b.

Let the length of the two equal sides of ΔABC be a.

Draw $AD \perp BC$.



Now, in ΔADC, by applying the Pythagoras theorem, we have:

$$AD = \sqrt{a^2 - \frac{b^2}{4}}$$

$$\therefore \text{ Area of triangle } (A) = \frac{1}{2} b \sqrt{a^2 - \frac{b^2}{4}}$$

The rate of change of the area with respect to time (t) is given by,

$$\frac{dA}{dt} = \frac{1}{2} b \cdot \frac{2a}{2\sqrt{a^2 - \frac{b^2}{4}}} \frac{da}{dt} = \frac{ab}{\sqrt{4a^2 - b^2}} \frac{da}{dt}$$

It is given that the two equal sides of the triangle are decreasing at the rate of 3 cm per second.

$$\therefore \frac{da}{dt} = -3 \text{ cm/s}$$

$$\therefore \frac{dA}{dt} = \frac{-3ab}{\sqrt{4a^2 - b^2}}$$

Then, when $a = b$, we have:

$$\frac{dA}{dt} = \frac{-3b^2}{\sqrt{4b^2 - b^2}} = \frac{-3b^2}{\sqrt{3b^2}} = -\sqrt{3}b$$

Hence, if the two equal sides are equal to the base, then the area of the triangle is decreasing at the rate of $\sqrt{3}\, b \text{ cm}^2/\text{s}$.

4. Find the equation of the normal to curve $x^2 = 4y$ which passes through the point (1, 2).

ANS :

The equation of the given curve is $y^2 = 4x$.

Differentiating with respect to x, we have:

$$2y\frac{dy}{dx} = 4$$

$$\Rightarrow \frac{dy}{dx} = \frac{4}{2y} = \frac{2}{y}$$

$$\therefore \left.\frac{dy}{dx}\right]_{(1,2)} = \frac{2}{2} = 1$$

Now, the slope of the normal at point (1, 2) is $\dfrac{-1}{\left.\frac{dy}{dx}\right]_{(1,2)}} = \dfrac{-1}{1} = -1.$

$\therefore$Equation of the normal at (1, 2) is $y - 2 = -1(x - 1)$.

$\Rightarrow y - 2 = -x + 1$

$\Rightarrow x + y - 3 = 0$

5. Show that the normal at any point θ to the curve
$x = a\cos\theta + a\theta\sin\theta$, $y = a\sin\theta - a\theta\cos\theta$
is at a constant distance from the origin.

ANS :

We have $x = a\cos\theta + a\theta\sin\theta$.

$$\therefore \frac{dx}{d\theta} = -a\sin\theta + a\sin\theta + a\theta\cos\theta = a\theta\cos\theta$$

$$y = a\sin\theta - a\theta\cos\theta$$

$$\therefore \frac{dy}{d\theta} = a\cos\theta - a\cos\theta + a\theta\sin\theta = a\theta\sin\theta$$

$$\therefore \frac{dy}{dx} = \frac{dy}{d\theta}\cdot\frac{d\theta}{dx} = \frac{a\theta\sin\theta}{a\theta\cos\theta} = \tan\theta$$

$\therefore$ Slope of the normal at any point θ is $-\frac{1}{\tan\theta}$.

The equation of the normal at a given point (x, y) is given by,

$$y - a\sin\theta + a\theta\cos\theta = \frac{-1}{\tan\theta}\left(x - a\cos\theta - a\theta\sin\theta\right)$$

$$\Rightarrow y\sin\theta - a\sin^2\theta + a\theta\sin\theta\cos\theta = -x\cos\theta + a\cos^2\theta + a\theta\sin\theta\cos\theta$$

$$\Rightarrow x\cos\theta + y\sin\theta - a\left(\sin^2\theta + \cos^2\theta\right) = 0$$

$$\Rightarrow x\cos\theta + y\sin\theta - a = 0$$

Now, the perpendicular distance of the normal from the origin is

$\frac{|-a|}{\sqrt{\cos^2\theta + \sin^2\theta}} = \frac{|-a|}{\sqrt{1}} = |-a|$, which is independent of θ.

Hence, the perpendicular distance of the normal from the origin is constant.

6. Find the intervals in which the function f given by

$$f(x)=\frac{4\sin x-2x-x\cos x}{2+\cos x}$$

is (i) increasing (ii) decreasing.

ANS :

$$f(x)=\frac{4\sin x-2x-x\cos x}{2+\cos x}$$

$$\therefore f'(x)=\frac{(2+\cos x)(4\cos x-2-\cos x+x\sin x)-(4\sin x-2x-x\cos x)(-\sin x)}{(2+\cos x)^2}$$

$$=\frac{(2+\cos x)(3\cos x-2+x\sin x)+\sin x(4\sin x-2x-x\cos x)}{(2+\cos x)^2}$$

$$=\frac{6\cos x-4+2x\sin x+3\cos^2 x-2\cos x+x\sin x\cos x+4\sin^2 x-2x\sin x-x\sin x\cos x}{(2+\cos x)^2}$$ Now, $f'(x)=0$

$$=\frac{4\cos x-4+3\cos^2 x+4\sin^2 x}{(2+\cos x)^2}$$

$$=\frac{4\cos x-4+3\cos^2 x+4-4\cos^2 x}{(2+\cos x)^2}$$

$$=\frac{4\cos x-\cos^2 x}{(2+\cos x)^2}=\frac{\cos x(4-\cos x)}{(2+\cos x)^2}$$

$\Rightarrow \cos x=0$ or $\cos x=4$

But, $\cos x\neq 4$

$\therefore \cos x=0$

$\Rightarrow x=\frac{\pi}{2},\frac{3\pi}{2}$

Now, $x=\frac{\pi}{2}$ and $x=\frac{3\pi}{2}$ divides $(0, 2\pi)$ into three disjoint intervals i.e.,

$\left(0,\frac{\pi}{2}\right),\left(\frac{\pi}{2},\frac{3\pi}{2}\right)$, and $\left(\frac{3\pi}{2},2\pi\right)$.

In intervals $\left(0,\frac{\pi}{2}\right)$ and $\left(\frac{3\pi}{2},2\pi\right)$, $f'(x)>0$.

Thus, $f(x)$ is increasing for $0<x<\frac{x}{2}$ and $\frac{3\pi}{2}<x<2\pi$.

In the interval $\left(\frac{\pi}{2},\frac{3\pi}{2}\right)$, $f'(x)<0$.

Thus, $f(x)$ is decreasing for $\frac{\pi}{2}<x<\frac{3\pi}{2}$.

7. Find the intervals in which the function f given by $f(x) = x^3 + \frac{1}{x^3}$, $x \neq 0$ is

(i) increasing (ii) decreasing.

ANS :

$$f(x) = x^3 + \frac{1}{x^3}$$

$$\therefore f'(x) = 3x^2 - \frac{3}{x^4} = \frac{3x^6 - 3}{x^4}$$

Then, $f'(x) = 0 \Rightarrow 3x^6 - 3 = 0 \Rightarrow x^6 = 1 \Rightarrow x = \pm 1$

Now, the points $x = 1$ and $x = -1$ divide the real line into three disjoint intervals i.e., $(-\infty, -1), (-1, 1)$, and $(1, \infty)$.

In intervals $(-\infty, -1)$ and $(1, \infty)$ i.e., when $x < -1$ and $x > 1$, $f'(x) > 0$.

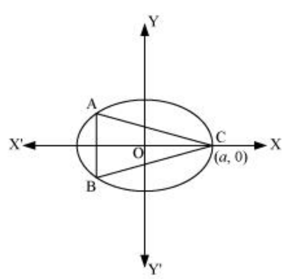
Thus, when $x < -1$ and $x > 1$, f is increasing.

In interval $(-1, 1)$ i.e., when $-1 < x < 1$, $f'(x) < 0$.

Thus, when $-1 < x < 1$, f is decreasing.

8. Find the maximum area of an isosceles triangle inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with its vertex at one end of the major axis.

ANS :



The given ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Let the major axis be along the x-axis.

Let ABC be the triangle inscribed in the ellipse where vertex C is at $(a, 0)$.

Since the ellipse is symmetrical with respect to the x-axis and y-axis, we can assume the coordinates of A to be $(-x_1, y_1)$ and the coordinates of B to be $(-x_1, -y_1)$.

Now, we have $y_1 = \pm\frac{b}{a}\sqrt{a^2 - x_1^2}$.

$\therefore$Coordinates of A are $\left(-x_1, \frac{b}{a}\sqrt{a^2 - x_1^2}\right)$ and the coordinates of B are $\left(x_1, -\frac{b}{a}\sqrt{a^2 - x_1^2}\right)$.

As the point (x_1, y_1) lies on the ellipse, the area of triangle ABC (A) is given by,

$$A=\frac{1}{2}\left|a\left(\frac{2b}{a}\sqrt{a^2-x_1^2}\right)+(-x_1)\left(-\frac{b}{a}\sqrt{a^2-x_1^2}\right)+(-x_1)\left(-\frac{b}{a}\sqrt{a^2-x_1^2}\right)\right|$$

$$\Rightarrow\ A=b\sqrt{a^2-x_1^2}+x_1\frac{b}{a}\sqrt{a^2-x_1^2} \qquad ...(1)$$

$$\therefore \frac{dA}{dx_1}=\frac{-2x_1b}{2\sqrt{a^2-x_1^2}}+\frac{b}{a}\sqrt{a^2-x_1^2}-\frac{2bx_1^2}{a2\sqrt{a^2-x_1^2}}$$

$$=\frac{b}{a\sqrt{a^2-x_1^2}}\left[-x_1a+\left(a^2-x_1^2\right)-x_1^2\right]$$

$$=\frac{b\left(-2x_1^2-x_1a+a^2\right)}{a\sqrt{a^2-x_1^2}}$$

Now, $\frac{dA}{dx_1}=0$

$$\Rightarrow -2x_1^2-x_1a+a^2=0$$

$$\Rightarrow x_1=\frac{a\pm\sqrt{a^2-4(-2)\left(a^2\right)}}{2(-2)}$$

$$=\frac{a\pm\sqrt{9a^2}}{-4}$$

$$=\frac{a\pm 3a}{-4}$$

$$\Rightarrow x_1=-a,\ \frac{a}{2}$$

But, x_1 cannot be equal to a.

$$\therefore x_1=\frac{a}{2}\Rightarrow y_1=\frac{b}{a}\sqrt{a^2-\frac{a^2}{4}}=\frac{ba}{2a}\sqrt{3}=\frac{\sqrt{3}b}{2}$$

Now, $$\frac{d^2A}{dx_1^2}=\frac{b}{a}\left\{\frac{\sqrt{a^2-x_1^2}\left(-4x_1-a\right)-\left(-2x_1^2-x_1a+a^2\right)\frac{\left(-2x_1\right)}{2\sqrt{a^2-x_1^2}}}{a^2-x_1^2}\right\}$$

$$=\frac{b}{a}\left\{\frac{\left(a^2-x_1^2\right)\left(-4x_1-a\right)+x_1\left(-2x_1^2-x_1a+a^2\right)}{\left(a^2-x_1^2\right)^{\frac{3}{2}}}\right\}$$

$$=\frac{b}{a}\left\{\frac{2x^3-3a^2x-a^3}{\left(a^2-x_1^2\right)^{\frac{3}{2}}}\right\}$$

Also, when $x_1=\frac{a}{2}$, then

$$\frac{d^2A}{dx_1^2}=\frac{b}{a}\left\{\frac{2\frac{a^3}{8}-3\frac{a^3}{2}-a^3}{\left(\frac{3a^2}{4}\right)^{\frac{3}{2}}}\right\}=\frac{b}{a}\left\{\frac{\frac{a^3}{4}-\frac{3}{2}a^3-a^3}{\left(\frac{3a^2}{4}\right)^{\frac{3}{2}}}\right\}$$

$$=-\frac{b}{a}\left\{\frac{\frac{9}{4}a^3}{\left(\frac{3a^2}{4}\right)^{\frac{3}{2}}}\right\}<0$$

Thus, the area is the maximum when $x_1=\frac{a}{2}$.

∴ Maximum area of the triangle is given by,

$$A=b\sqrt{a^2-\frac{a^2}{4}}+\left(\frac{a}{2}\right)\frac{b}{a}\sqrt{a^2-\frac{a^2}{4}}$$

$$=ab\frac{\sqrt{3}}{2}+\left(\frac{a}{2}\right)\frac{b}{a}\times\frac{a\sqrt{3}}{2}$$

$$=\frac{ab\sqrt{3}}{2}+\frac{ab\sqrt{3}}{4}=\frac{3\sqrt{3}}{4}ab$$

9. A tank with rectangular base and rectangular sides, open at the top is to be constructed so that its depth is 2 m and volume is 8 m^3. If building of tank costs Rs 70 per sq metres for the base and Rs 45 per square metre for sides. What is the cost of least expensive tank?

ANS :

Let l, b, and h represent the length, breadth, and height of the tank respectively.

Then, we have height $(h) = 2$ m

Volume of the tank $= 8\text{m}^3$

Volume of the tank $= l \times b \times h$

$\therefore 8 = l \times b \times 2$

$\Rightarrow lb = 4 \Rightarrow b = \frac{4}{l}$

Now, area of the base $= lb = 4$

Area of the 4 walls $(A) = 2h\,(l + b)$

$$\therefore A = 4\left(l + \frac{4}{l}\right)$$

$$\Rightarrow \frac{dA}{dl} = 4\left(1 - \frac{4}{l^2}\right)$$

Now, $\frac{dA}{dl} = 0$

$$\Rightarrow 1 - \frac{4}{l^2} = 0$$

$$\Rightarrow l^2 = 4$$

$$\Rightarrow l = \pm 2$$

However, the length cannot be negative.

Therefore, we have $l = 4$.

$\therefore b = \frac{4}{l} = \frac{4}{2} = 2$

Now, $\frac{d^2A}{dl^2} = \frac{32}{l^3}$

When $l = 2$, $\frac{d^2A}{dl^2} = \frac{32}{8} = 4 > 0$.

Thus, by second derivative test, the area is the minimum when $l = 2$.

We have $l = b = h = 2$.

$\therefore$Cost of building the base = Rs 70 × (lb) = Rs 70 (4) = Rs 280

Cost of building the walls = Rs 2h (l + b) × 45 = Rs 90 (2) (2 + 2)

= Rs 8 (90) = Rs 720

Required total cost = Rs (280 + 720) = Rs 1000

Hence, the total cost of the tank will be Rs 1000.

10. The sum of the perimeter of a circle and square is k, where k is some constant. Prove that the sum of their areas is least when the side of square is double the radius of the circle.

ANS :

Let r be the radius of the circle and a be the side of the square.

Then, we have:

$$2\pi r + 4a = k \text{ (where } k \text{ is constant)}$$
$$\Rightarrow a = \frac{k - 2\pi r}{4}$$

The sum of the areas of the circle and the square (A) is given by,

$$A = \pi r^2 + a^2 = \pi r^2 + \frac{(k - 2\pi r)^2}{16}$$
$$\therefore \frac{dA}{dr} = 2\pi r + \frac{2(k - 2\pi r)(-2\pi)}{16} = 2\pi r - \frac{\pi(k - 2\pi r)}{4}$$

$$\text{Now, } \frac{dA}{dr} = 0$$
$$\Rightarrow 2\pi r = \frac{\pi(k - 2\pi r)}{4}$$
$$8r = k - 2\pi r$$
$$\Rightarrow (8 + 2\pi) r = k$$
$$\Rightarrow r = \frac{k}{8 + 2\pi} = \frac{k}{2(4 + \pi)}$$
$$\text{Now, } \frac{d^2 A}{dr^2} = 2\pi + \frac{\pi^2}{2} > 0$$
$$\therefore \text{When } r = \frac{k}{2(4+\pi)}, \ \frac{d^2 A}{dr^2} > 0.$$

$\therefore$ The sum of the areas is least when $r = \frac{k}{2(4+\pi)}$.

When $r = \frac{k}{2(4+\pi)}$, $a = \frac{k - 2\pi\left[\frac{k}{2(4+\pi)}\right]}{4} = \frac{k(4+\pi) - \pi k}{4 \cdot 4(4+\pi)} = \frac{4k}{4(4+\pi)} = \frac{k}{4+\pi} = 2r.$

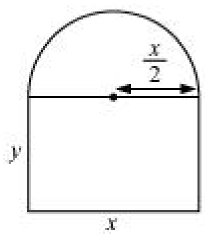
Hence, it has been proved that the sum of their areas is least when the side of the square is double the radius of the circle.

11. A window is in the form of a rectangle surmounted by a semicircular opening. The total perimeter of the window is 10 m. Find the dimensions of the window to admit maximum light through the whole opening.

ANS :

Let x and y be the length and breadth of the rectangular window.

Radius of the semicircular opening $= \frac{x}{2}$



It is given that the perimeter of the window is 10 m.

$$\therefore x + 2y + \frac{\pi x}{2} = 10$$

$$\Rightarrow x\left(1 + \frac{\pi}{2}\right) + 2y = 10$$

$$\Rightarrow 2y = 10 - x\left(1 + \frac{\pi}{2}\right)$$

$$\Rightarrow y = 5 - x\left(\frac{1}{2} + \frac{\pi}{4}\right)$$

$\therefore$Area of the window (A) is given by,

$$A = xy + \frac{\pi}{2}\left(\frac{x}{2}\right)^2$$

$$= x\left[5 - x\left(\frac{1}{2} + \frac{\pi}{4}\right)\right] + \frac{\pi}{8}x^2$$

$$= 5x - x^2\left(\frac{1}{2} + \frac{\pi}{4}\right) + \frac{\pi}{8}x^2$$

$$\therefore \frac{dA}{dx} = 5 - 2x\left(\frac{1}{2} + \frac{\pi}{4}\right) + \frac{\pi}{4}x$$

$$= 5 - x\left(1 + \frac{\pi}{2}\right) + \frac{\pi}{4}x$$

$$\therefore \frac{d^2A}{dx^2} = -\left(1 + \frac{\pi}{2}\right) + \frac{\pi}{4} = -1 - \frac{\pi}{4}$$

Now, $\frac{dA}{dx} = 0$

$$\Rightarrow 5 - x\left(1 + \frac{\pi}{2}\right) + \frac{\pi}{4}x = 0$$

$$\Rightarrow 5 - x - \frac{\pi}{4}x = 0$$

$$\Rightarrow x\left(1 + \frac{\pi}{4}\right) = 5$$

$$\Rightarrow x = \frac{5}{\left(1 + \frac{\pi}{4}\right)} = \frac{20}{\pi + 4}$$

Thus, when $x = \frac{20}{\pi + 4}$ then $\frac{d^2A}{dx^2} < 0$.

Therefore, by second derivative test, the area is the maximum when length $x = \frac{20}{\pi + 4}$ m.

Now,

$$y = 5 - \frac{20}{\pi + 4}\left(\frac{2 + \pi}{4}\right) = 5 - \frac{5(2 + \pi)}{\pi + 4} = \frac{10}{\pi + 4} \text{ m}$$

Hence, the required dimensions of the window to admit maximum light is given by length $= \frac{20}{\pi + 4}$ m and breadth $= \frac{10}{\pi + 4}$ m.

12. A point on the hypotenuse of a triangle is at distance a and b from the sides of the triangle.

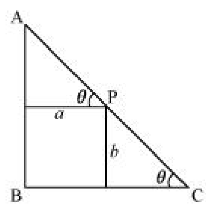
Show that the maximum length of the hypotenuse is $\left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{3}{2}}$.

ANS :

Let ΔABC be right-angled at B. Let AB $= x$ and BC $= y$.

Let P be a point on the hypotenuse of the triangle such that P is at a distance of a and b from the sides AB and BC respectively.

Let $\angle C = \theta$.



We have,

$$AC = \sqrt{x^2 + y^2}$$

Now,

$PC = b \operatorname{cosec} \theta$

And, $AP = a \sec \theta$

$\therefore AC = AP + PC$

$\Rightarrow AC = b \operatorname{cosec} \theta + a \sec \theta \ldots (1)$

$$\therefore \frac{d(\text{AC})}{d\theta} = -b\text{cosec}\theta\cot\theta + a\sec\theta\tan\theta$$

$$\therefore \frac{d(\text{AC})}{d\theta} = 0$$

$$\Rightarrow a\sec\theta\tan\theta = b\text{cosec}\theta\cot\theta$$

$$\Rightarrow \frac{a}{\cos\theta}\cdot\frac{\sin\theta}{\cos\theta} = \frac{b}{\sin\theta}\frac{\cos\theta}{\sin\theta}$$

$$\Rightarrow a\sin^3\theta = b\cos^3\theta$$

$$\Rightarrow (a)^{\frac{1}{3}}\sin\theta = (b)^{\frac{1}{3}}\cos\theta$$

$$\Rightarrow \tan\theta = \left(\frac{b}{a}\right)^{\frac{1}{3}}$$

$$\therefore \sin\theta = \frac{(b)^{\frac{1}{3}}}{\sqrt{a^{\frac{2}{3}} + b^{\frac{2}{3}}}} \text{ and } \cos\theta = \frac{(a)^{\frac{1}{3}}}{\sqrt{a^{\frac{2}{3}} + b^{\frac{2}{3}}}} \qquad \ldots(2)$$

It can be clearly shown that $\frac{d^2(\text{AC})}{d\theta^2} < 0$ when $\tan\theta = \left(\frac{b}{a}\right)^{\frac{1}{3}}$.

Therefore, by second derivative test, the length of the hypotenuse is the maximum when $\tan\theta = \left(\frac{b}{a}\right)^{\frac{1}{3}}$.

Now, when $\tan\theta = \left(\frac{b}{a}\right)^{\frac{1}{3}}$, we have:

$$\text{AC} = \frac{b\sqrt{a^{\frac{2}{3}} + b^{\frac{2}{3}}}}{b^{\frac{1}{3}}} + \frac{a\sqrt{a^{\frac{2}{3}} + b^{\frac{2}{3}}}}{a^{\frac{1}{3}}} \qquad [\text{Using (1) and (2)}]$$

$$= \sqrt{a^{\frac{2}{3}} + b^{\frac{2}{3}}}\left(b^{\frac{2}{3}} + a^{\frac{2}{3}}\right)$$

$$= \left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{3}{2}}$$

Hence, the maximum length of the hypotenuses is $\left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{3}{2}}$.

13. Find the points at which the function f given by $f(x) = (x-2)^4 (x+1)^3$ has

(i) local maxima (ii) local minima

(iii) point of inflexion

ANS :

The given function is $f(x)=(x-2)^4(x+1)^3$.

$$\therefore f'(x)=4(x-2)^3(x+1)^3+3(x+1)^2(x-2)^4$$
$$=(x-2)^3(x+1)^2\left[4(x+1)+3(x-2)\right]$$
$$=(x-2)^3(x+1)^2(7x-2)$$

Now, $f'(x)=0 \Rightarrow x=-1$ and $x=\frac{2}{7}$ or $x=2$

Now, for values of x close to $\frac{2}{7}$ and to the left of $\frac{2}{7}$, $f'(x)>0$. Also, for values of x close to $\frac{2}{7}$ and to the right of $\frac{2}{7}$, $f'(x)<0$.

Thus, $x=\frac{2}{7}$ is the point of local maxima.

Now, for values of x close to 2 and to the left of 2, $f'(x)<0$. Also, for values of x close to 2 and to the right of 2, $f'(x)>0$.

Thus, $x=2$ is the point of local minima.

Now, as the value of x varies through -1, $f'(x)$ does not changes its sign.

Thus, $x=-1$ is the point of inflexion.

14. Find the absolute maximum and minimum values of the function f given by

$$f(x) = \cos^2 x + \sin x, x \in [0, \pi]$$

ANS :

$f(x) = \cos^2 x + \sin x$

$f'(x) = 2\cos x(-\sin x) + \cos x$

$= -2\sin x \cos x + \cos x$

Now, $f'(x) = 0$

$\Rightarrow 2\sin x \cos x = \cos x \Rightarrow \cos x(2\sin x - 1) = 0$

$\Rightarrow \sin x = \frac{1}{2}$ or $\cos x = 0$

$\Rightarrow x = \frac{\pi}{6}$, or $\frac{\pi}{2}$ as $x \in [0, \pi]$

Now, evaluating the value of f at critical points $x = \frac{\pi}{2}$ and $x = \frac{\pi}{6}$ and at the end points of the interval $[0, \pi]$ (i.e., at $x = 0$ and $x = \pi$), we have:

$f\left(\frac{\pi}{6}\right) = \cos^2 \frac{\pi}{6} + \sin \frac{\pi}{6} = \left(\frac{\sqrt{3}}{2}\right)^2 + \frac{1}{2} = \frac{5}{4}$

$f(0) = \cos^2 0 + \sin 0 = 1 + 0 = 1$

$f(\pi) = \cos^2 \pi + \sin \pi = (-1)^2 + 0 = 1$

$f\left(\frac{\pi}{2}\right) = \cos^2 \frac{\pi}{2} + \sin \frac{\pi}{2} = 0 + 1 = 1$

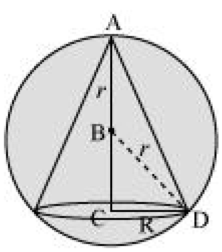
Hence, the absolute maximum value of f is $\frac{5}{4}$ occurring at $x = \frac{\pi}{6}$ and the absolute minimum value of f is 1 occurring at $x = 0, \frac{\pi}{2}$, and π.

15. Show that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius r is $\frac{4r}{3}$.

ANS :

A sphere of fixed radius (r) is given.

Let R and h be the radius and the height of the cone respectively.



The volume (V) of the cone is given by,

$V = \frac{1}{3}\pi R^2 h$

Now, from the right triangle BCD, we have:

$$BC = \sqrt{r^2 - R^2}$$

$$\therefore h = r + \sqrt{r^2 - R^2}$$

$$\therefore V = \frac{1}{3}\pi R^2\left(r + \sqrt{r^2 - R^2}\right) = \frac{1}{3}\pi R^2 r + \frac{1}{3}\pi R^2\sqrt{r^2 - R^2}$$

$$\therefore \frac{dV}{dR} = \frac{2}{3}\pi Rr + \frac{2\pi}{3}R\sqrt{r^2 - R^2} + \frac{R^2}{3}\cdot\frac{(-2R)}{2\sqrt{r^2 - R^2}}$$

$$= \frac{2}{3}\pi Rr + \frac{2\pi}{3}R\sqrt{r^2 - R^2} - \frac{R^3}{3\sqrt{r^2 - R^2}}$$

$$= \frac{2}{3}\pi Rr + \frac{2\pi R\left(r^2 - R^2\right) - \pi R^3}{3\sqrt{r^2 - R^2}}$$

$$= \frac{2}{3}\pi Rr + \frac{2\pi Rr^2 - 3\pi R^3}{3\sqrt{r^2 - R^2}}$$

$$\text{Now,} \frac{dV}{dR^2} = 0$$

$$\Rightarrow \frac{2\pi rR}{3} = \frac{3\pi R^3 - 2\pi Rr^2}{3\sqrt{r^2 - R^2}}$$

$$\Rightarrow 2r\sqrt{r^2 - R^2} = 3R^2 - 2r^2$$

$$\Rightarrow 4r^2\left(r^2 - R^2\right) = \left(3R^2 - 2r^2\right)^2$$

$$\Rightarrow 4r^4 - 4r^2R^2 = 9R^4 + 4r^4 - 12R^2r^2$$

$$\Rightarrow 9R^4 - 8r^2R^2 = 0$$

$$\Rightarrow 9R^2 = 8r^2$$

$$\Rightarrow R^2 = \frac{8r^2}{9}$$

$$\text{Now,} \frac{d^2V}{dR^2} = \frac{2\pi r}{3} + \frac{3\sqrt{r^2 - R^2}\left(2\pi r^2 - 9\pi R^2\right) - \left(2\pi Rr^2 - 3\pi R^3\right)(-6R)\frac{1}{2\sqrt{r^2 - R^2}}}{9\left(r^2 - R^2\right)}$$

$$= \frac{2\pi r}{3} + \frac{3\sqrt{r^2 - R^2}\left(2\pi r^2 - 9\pi R^2\right) + \left(2\pi Rr^2 - 3\pi R^3\right)(3R)\frac{1}{2\sqrt{r^2 - R^2}}}{9\left(r^2 - R^2\right)}$$

Now, when $R^2 = \frac{8r^2}{9}$, it can be shown that $\frac{d^2V}{dR^2} < 0$.

$\therefore$ The volume is the maximum when $R^2 = \frac{8r^2}{9}$.

When $R^2 = \frac{8r^2}{9}$, height of the cone $= r + \sqrt{r^2 - \frac{8r^2}{9}} = r + \sqrt{\frac{r^2}{9}} = r + \frac{r}{3} = \frac{4r}{3}$.

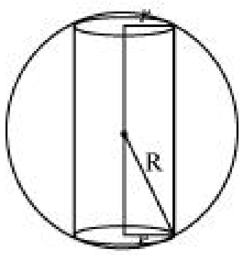
Hence, it can be seen that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius r is $\frac{4r}{3}$.

17. Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius R is $\frac{2R}{\sqrt{3}}$. Also find the maximum volume.

ANS :

A sphere of fixed radius (R) is given.

Let r and h be the radius and the height of the cylinder respectively.



From the given figure, we have $h = 2\sqrt{R^2 - r^2}$.

The volume (V) of the cylinder is given by,

$$V = \pi r^2 h = 2\pi r^2 \sqrt{R^2 - r^2}$$

$$\therefore \frac{dV}{dr} = 4\pi r\sqrt{R^2 - r^2} + \frac{2\pi r^2(-2r)}{2\sqrt{R^2 - r^2}}$$

$$= 4\pi r\sqrt{R^2 - r^2} - \frac{2\pi r^3}{\sqrt{R^2 - r^2}}$$

$$= \frac{4\pi r(R^2 - r^2) - 2\pi r^3}{\sqrt{R^2 - r^2}}$$

$$= \frac{4\pi rR^2 - 6\pi r^3}{\sqrt{R^2 - r^2}}$$

Now, $\frac{dV}{dr} = 0 \Rightarrow 4\pi rR^2 - 6\pi r^3 = 0$

$$\Rightarrow r^2 = \frac{2R^2}{3}$$

Now, $\dfrac{d^2V}{dr^2} = \dfrac{\sqrt{R^2 - r^2}\left(4\pi R^2 - 18\pi r^2\right) - \left(4\pi r R^2 - 6\pi r^3\right)\dfrac{(-2r)}{2\sqrt{R^2 - r^2}}}{\left(R^2 - r^2\right)}$

$$= \frac{\left(R^2 - r^2\right)\left(4\pi R^2 - 18\pi r^2\right) + r\left(4\pi r R^2 - 6\pi r^3\right)}{\left(R^2 - r^2\right)^{\frac{3}{2}}}$$

$$= \frac{4\pi R^4 - 22\pi r^2 R^2 + 12\pi r^4 + 4\pi r^2 R^2}{\left(R^2 - r^2\right)^{\frac{3}{2}}}$$

Now, it can be observed that at $r^2 = \dfrac{2R^2}{3}, \dfrac{d^2V}{dr^2} < 0$.

$\therefore$The volume is the maximum when $r^2 = \dfrac{2R^2}{3}$.

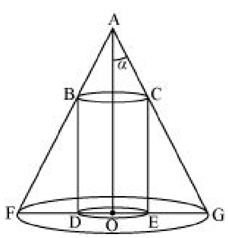
When $r^2 = \dfrac{2R^2}{3}$, the height of the cylinder is $2\sqrt{R^2 - \dfrac{2R^2}{3}} = 2\sqrt{\dfrac{R^2}{3}} = \dfrac{2R}{\sqrt{3}}$.

Hence, the volume of the cylinder is the maximum when the height of the cylinder is $\dfrac{2R}{\sqrt{3}}$.

18. **Show that height of the cylinder of greatest volume which can be inscribed in a right circular cone of height h and semi vertical angle α is one-third that of the cone and the greatest volume of cylinder is $\dfrac{4}{27}\pi h^3 \tan^2 \alpha$.**

ANS :

The given right circular cone of fixed height (h) and semi-vertical angle (α) can be drawn as:



Here, a cylinder of radius R and height H is inscribed in the cone.

Then, $\angle$GAO $= \alpha$, OG $= r$, OA $= h$, OE $= R$, and CE $= H$.

We have,

$r = h \tan \alpha$

Now, since ΔAOG is similar to ΔCEG, we have:

$$\frac{AO}{OG} = \frac{CE}{EG}$$

$$\Rightarrow \frac{h}{r} = \frac{H}{r - R} \qquad [EG = OG - OE]$$

$$\Rightarrow H = \frac{h}{r}(r - R) = \frac{h}{h\tan\alpha}(h\tan\alpha - R) = \frac{1}{\tan\alpha}(h\tan\alpha - R)$$

Now, the volume (V) of the cylinder is given by,

$$V = \pi R^2 H = \frac{\pi R^2}{\tan\alpha}(h\tan\alpha - R) = \pi R^2 h - \frac{\pi R^3}{\tan\alpha}$$

$$\therefore \frac{dV}{dR} = 2\pi Rh - \frac{3\pi R^2}{\tan\alpha}$$

Now, $\frac{dV}{dR} = 0$

$$\Rightarrow 2\pi Rh = \frac{3\pi R^2}{\tan\alpha}$$

$$\Rightarrow 2h\tan\alpha = 3R$$

$$\Rightarrow R = \frac{2h}{3}\tan\alpha$$

Now, $\frac{d^2V}{dR^2} = 2\pi h - \frac{6\pi R}{\tan\alpha}$

And, for $R = \frac{2h}{3}\tan\alpha$, we have:

$$\frac{d^2V}{dR^2} = 2\pi h - \frac{6\pi}{\tan\alpha}\left(\frac{2h}{3}\tan\alpha\right) = 2\pi h - 4\pi h = -2\pi h < 0$$

∴By second derivative test, the volume of the cylinder is the greatest when

$R = \frac{2h}{3}\tan\alpha.$

When $R = \frac{2h}{3}\tan\alpha,\ H = \frac{1}{\tan\alpha}\left(h\tan\alpha - \frac{2h}{3}\tan\alpha\right) = \frac{1}{\tan\alpha}\left(\frac{h\tan\alpha}{3}\right) = \frac{h}{3}.$

Thus, the height of the cylinder is one-third the height of the cone when the volume of the cylinder is the greatest.

Now, the maximum volume of the cylinder can be obtained as:

$$\pi\left(\frac{2h}{3}\tan\alpha\right)^2\left(\frac{h}{3}\right) = \pi\left(\frac{4h^2}{9}\tan^2\alpha\right)\left(\frac{h}{3}\right) = \frac{4}{27}\pi h^3\tan^2\alpha$$

Hence, the given result is proved.

19. A cylindrical tank of radius 10 m is being filled with wheat at the rate of 314 cubic metre per hour. Then the depth of the wheat is increasing at the rate of

(A) $1 \text{ m}^3/\text{h}$ (B) $0.1 \text{ m}^3/\text{h}$

(C) $1.1 \text{ m}^3/\text{h}$ (D) $0.5 \text{ m}^3/\text{h}$

ANS :

Let r be the radius of the cylinder.

Then, volume (V) of the cylinder is given by,

$$V = \pi(\text{radius})^2 \times \text{height}$$
$$= \pi(10)^2 h \qquad (\text{radius} = 10 \text{ m})$$
$$= 100\pi h$$

Differentiating with respect to time t, we have:

$$\frac{dV}{dt} = 100\pi \frac{dh}{dt}$$

The tank is being filled with wheat at the rate of 314 cubic metres per hour.

$$\therefore \frac{dV}{dt} = 314 \text{ m}^3/\text{h}$$

Thus, we have:

$$314 = 100\pi \frac{dh}{dt}$$
$$\Rightarrow \frac{dh}{dt} = \frac{314}{100(3.14)} = \frac{314}{314} = 1$$

Hence, the depth of wheat is increasing at the rate of 1 m/h.

The correct answer is A.

20. The slope of the tangent to the curve $x = t^2 + 3t - 8, y = 2t^2 - 2t - 5$ at the point (2,– 1) is

(A) $\frac{22}{7}$ (B) $\frac{6}{7}$ (C) $\frac{7}{6}$ (D) $\frac{-6}{7}$

ANS :

The given curve is $x = t^2 + 3t - 8$ and $y = 2t^2 - 2t - 5$.

$$\therefore \frac{dx}{dt} = 2t + 3 \text{ and } \frac{dy}{dt} = 4t - 2$$

$$\therefore \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{4t-2}{2t+3}$$

The given point is $(2, -1)$.

At $x = 2$, we have:

$$t^2 + 3t - 8 = 2$$
$$\Rightarrow t^2 + 3t - 10 = 0$$
$$\Rightarrow (t-2)(t+5) = 0$$
$$\Rightarrow t = 2 \text{ or } t = -5$$

At $y = -1$, we have:

$$2t^2 - 2t - 5 = -1$$
$$\Rightarrow 2t^2 - 2t - 4 = 0$$
$$\Rightarrow 2(t^2 - t - 2) = 0$$
$$\Rightarrow (t-2)(t+1) = 0$$
$$\Rightarrow t = 2 \text{ or } t = -1$$

The common value of t is 2.

Hence, the slope of the tangent to the given curve at point $(2, -1)$ is

$$\left.\frac{dy}{dx}\right]_{t=2} = \frac{4(2)-2}{2(2)+3} = \frac{8-2}{4+3} = \frac{6}{7}.$$

The correct answer is B

21. The line $y = mx + 1$ is a tangent to the curve $y^2 = 4x$ if the value of m is

(A) 1 (B) 2 (C) 3 (D) $\frac{1}{2}$

ANS :

The equation of the tangent to the given curve is $y = mx + 1$.

Now, substituting $y = mx + 1$ in $y^2 = 4x$, we get:

$$\Rightarrow (mx+1)^2 = 4x$$
$$\Rightarrow m^2x^2 + 1 + 2mx - 4x = 0$$
$$\Rightarrow m^2x^2 + x(2m-4) + 1 = 0 \qquad ...(i)$$

Since a tangent touches the curve at one point, the roots of equation (i) must be equal.

Therefore, we have:

Discriminant = 0

$$(2m-4)^2 - 4(m^2)(1) = 0$$
$$\Rightarrow 4m^2 + 16 - 16m - 4m^2 = 0$$
$$\Rightarrow 16 - 16m = 0$$
$$\Rightarrow m = 1$$

Hence, the required value of m is 1.

The correct answer is A.

22. The normal at the point (1,1) on the curve $2y + x^2 = 3$ is

(A) $x + y = 0$ (B) $x - y = 0$

(C) $x + y + 1 = 0$ (D) $x - y = 0$

ANS :

The equation of the given curve is $2y + x^2 = 3$.

Differentiating with respect to x, we have:

$$\frac{2dy}{dx} + 2x = 0$$
$$\Rightarrow \frac{dy}{dx} = -x$$
$$\therefore \left.\frac{dy}{dx}\right]_{(1,1)} = -1$$

The slope of the normal to the given curve at point (1, 1) is

$$\frac{-1}{\left.\frac{dy}{dx}\right]_{(1,1)}} = 1.$$

Hence, the equation of the normal to the given curve at (1, 1) is given as:

$$\Rightarrow y - 1 = 1(x - 1)$$
$$\Rightarrow y - 1 = x - 1$$
$$\Rightarrow x - y = 0$$

The correct answer is B.

23. The normal to the curve $x^2 = 4y$ passing $(1,2)$ is

(A) $x + y = 3$ (B) $x - y = 3$

(C) $x + y = 1$ (D) $x - y = 1$

ANS :

The equation of the given curve is $x^2 = 4y$.

Differentiating with respect to x, we have:

$$2x = 4 \cdot \frac{dy}{dx}$$
$$\Rightarrow \frac{dy}{dx} = \frac{x}{2}$$

The slope of the normal to the given curve at point (h, k) is given by

$$\frac{-1}{\left.\frac{dy}{dx}\right]_{(h,k)}} = -\frac{2}{h}$$

$\therefore$Equation of the normal at point (h, k) is given as:

$$y - k = \frac{-2}{h}(x - h)$$

Now, it is given that the normal passes through the point $(1, 2)$.

Therefore, we have:

$$2 - k = \frac{-2}{h}(1 - h) \text{ or } k = 2 + \frac{2}{h}(1 - h) \qquad \text{... (i)}$$

Since (h, k) lies on the curve $x^2 = 4y$, we have $h^2 = 4k$.

$$\Rightarrow k = \frac{h^2}{4}$$

From equation (i), we have:

$$\frac{h^2}{4} = 2 + \frac{2}{h}(1-h)$$

$$\Rightarrow \frac{h^3}{4} = 2h + 2 - 2h = 2$$

$$\Rightarrow h^3 = 8$$

$$\Rightarrow h = 2$$

$$\therefore k = \frac{h^2}{4} \Rightarrow k = 1$$

Hence, the equation of the normal is given as:

$$\Rightarrow y - 1 = \frac{-2}{2}(x-2)$$

$$\Rightarrow y - 1 = -(x-2)$$

$$\Rightarrow x + y = 3$$

The correct answer is A.

24. The points on the curve $9y^2 = x^3$, where the normal to the curve makes equal intercepts with the axes are

(A) $\left(4, \pm\frac{8}{3}\right)$ (B) $\left(4, \frac{-8}{3}\right)$

(C) $\left(4, \pm\frac{3}{8}\right)$ (D) $\left(\pm 4, \frac{3}{8}\right)$

ANS :

The equation of the given curve is $9y^2 = x^3$.

Differentiating with respect to x, we have:

$$9(2y)\frac{dy}{dx} = 3x^2$$
$$\Rightarrow \frac{dy}{dx} = \frac{x^2}{6y}$$

The slope of the normal to the given curve at point (x_1, y_1) is

$$\frac{-1}{\left.\frac{dy}{dx}\right]_{(x_1,y_1)}} = -\frac{6y_1}{x_1^2}.$$

$\therefore$ The equation of the normal to the curve at (x_1, y_1) is

$$y - y_1 = \frac{-6y_1}{x_1^2}(x - x_1).$$
$$\Rightarrow x_1^2 y - x_1^2 y_1 = -6xy_1 + 6x_1 y_1$$
$$\Rightarrow 6xy_1 + x_1^2 y = 6x_1 y_1 + x_1^2 y_1$$
$$\Rightarrow \frac{6xy_1}{6x_1 y_1 + x_1^2 y_1} + \frac{x_1^2 y}{6x_1 y_1 + x_1^2 y_1} = 1$$
$$\Rightarrow \frac{x}{\frac{x_1(6 + x_1)}{6}} + \frac{y}{\frac{y_1(6 + x_1)}{x_1}} = 1$$

It is given that the normal makes equal intercepts with the axes.

Therefore, We have:

$$\therefore \frac{x_1(6+x_1)}{6} = \frac{y_1(6+x_1)}{x_1}$$

$$\Rightarrow \frac{x_1}{6} = \frac{y_1}{x_1}$$

$$\Rightarrow x_1^2 = 6y_1 \qquad ...(i)$$

Also, the point (x_1, y_1) lies on the curve, so we have

$$9y_1^2 = x_1^3 \qquad ...(ii)$$

From (i) and (ii), we have:

$$9\left(\frac{x_1^2}{6}\right)^2 = x_1^3 \Rightarrow \frac{x_1^4}{4} = x_1^3 \Rightarrow x_1 = 4$$

From (ii), we have:

$$9y_1^2 = (4)^3 = 64$$

$$\Rightarrow y_1^2 = \frac{64}{9}$$

$$\Rightarrow y_1 = \pm\frac{8}{3}$$

Hence, the required points are $\left(4, \pm\frac{8}{3}\right)$.

The correct answer is A.