CLASS XII APPLICATION OF DERIVATIVES CHAPTER 6

EX. 6.3 SOLUTIONS

1. Find the slope of the tangent to the curve $y = 3x^4 - 4x$ at $x = 4$.

ANS :

The given curve is $y = 3x^4 - 4x$.

Then, the slope of the tangent to the given curve at $x = 4$ is given by,

$$\left.\frac{dy}{dx}\right]_{x=4} = 12x^3 - 4\Big]_{x=4} = 12(4)^3 - 4 = 12(64) - 4 = 764$$

2. Find the slope of the tangent to the curve $y = \frac{x-1}{x-2}$, $x \neq 2$ at $x = 10$.

ANS :

The given curve is $y = \frac{x-1}{x-2}$.

$$\therefore \frac{dy}{dx} = \frac{(x-2)(1)-(x-1)(1)}{(x-2)^2}$$

$$= \frac{x-2-x+1}{(x-2)^2} = \frac{-1}{(x-2)^2}$$

Thus, the slope of the tangent at $x = 10$ is given by,

$$\left.\frac{dy}{dx}\right]_{x=10} = \left.\frac{-1}{(x-2)^2}\right]_{x=10} = \frac{-1}{(10-2)^2} = \frac{-1}{64}$$

Hence, the slope of the tangent at $x = 10$ is $\frac{-1}{64}$.

3. Find the slope of the tangent to curve $y = x^3 - x + 1$ at the point whose x-coordinate is 2.

ANS :

The given curve is $y = x^3 - x + 1$.

$$\therefore \frac{dy}{dx} = 3x^2 - 1$$

The slope of the tangent to a curve at (x_0, y_0) is $\left.\frac{dy}{dx}\right]_{(x_0, y_0)}$.

It is given that $x_0 = 2$.

Hence, the slope of the tangent at the point where the x-coordinate is 2 is given by,

$$\left.\frac{dy}{dx}\right]_{x=2} = 3x^2 - 1\Big]_{x=2} = 3(2)^2 - 1 = 12 - 1 = 11$$

4. Find the slope of the tangent to the curve $y = x^3 - 3x + 2$ at the point whose x-coordinate is 3.

ANS :

The given curve is $y = x^3 - 3x + 2$.

$$\therefore \frac{dy}{dx} = 3x^2 - 3$$

The slope of the tangent to a curve at (x_0, y_0) is $\left.\frac{dy}{dx}\right]_{(x_0, y_0)}$.

Hence, the slope of the tangent at the point where the x-coordinate is 3 is given by,

$$\left.\frac{dy}{dx}\right]_{x=3} = 3x^2 - 3\Big]_{x=3} = 3(3)^2 - 3 = 27 - 3 = 24$$

5. Find the slope of the normal to the curve $x = a\cos^3\theta,\ y = a\sin^3\theta$ at $\theta = \frac{\pi}{4}$.

ANS :

It is given that $x = a\cos^3\theta$ and $y = a\sin^3\theta$.

$$\therefore \frac{dx}{d\theta} = 3a\cos^2\theta(-\sin\theta) = -3a\cos^2\theta\sin\theta$$

$$\frac{dy}{d\theta} = 3a\sin^2\theta(\cos\theta)$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta} = -\frac{\sin\theta}{\cos\theta} = -\tan\theta$$

Therefore, the slope of the tangent at $\theta = \frac{\pi}{4}$ is given by,

$$\left.\frac{dy}{dx}\right]_{\theta=\frac{\pi}{4}} = -\tan\theta\Big]_{\theta=\frac{\pi}{4}} = -\tan\frac{\pi}{4} = -1$$

Hence, the slope of the normal at $\theta = \frac{\pi}{4}$ is given by,

$$\frac{1}{\text{slope of the tangent at } \theta = \frac{\pi}{4}} = \frac{-1}{-1} = 1$$

6. Find the slope of the normal to the curve $x = 1 - a\sin\theta, y = b\cos^2\theta$ at $\theta - \frac{\pi}{2}$.

ANS :

It is given that $x = 1 - a\sin\theta$ and $y = b\cos^2\theta$.

$$\therefore \frac{dx}{d\theta} = -a\cos\theta \ and\ \frac{dy}{d\theta} = 2b\cos\theta(-\sin\theta) = -2b\sin\theta\cos\theta$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{-2b\sin\theta\cos\theta}{-a\cos\theta} = \frac{2b}{a}\sin\theta$$

Therefore, the slope of the tangent at $\theta = \frac{\pi}{2}$ is given by,

$$\left.\frac{dy}{dx}\right]_{\theta=\frac{\pi}{2}} = \left.\frac{2b}{a}\sin\theta\right]_{\theta=\frac{\pi}{2}} = \frac{2b}{a}\sin\frac{\pi}{2} = \frac{2b}{a}$$

Hence, the slope of the normal at $\theta = \frac{\pi}{2}$ is given by,

$$\frac{1}{\text{slope of the tangent at } \theta = \frac{\pi}{4}} = \frac{-1}{\left(\frac{2b}{a}\right)} = -\frac{a}{2b}$$

7. **Find points at which the tangent to the curve $y = x^3 - 3x^2 - 9x + 7$ is parallel to the x-axis.**

ANS :

The equation of the given curve is $y = x^3 - 3x^2 - 9x + 7$.

$$\therefore \frac{dy}{dx} = 3x^2 - 6x - 9$$

Now, the tangent is parallel to the x-axis if the slope of the tangent is zero.

$$\therefore 3x^2 - 6x - 9 = 0 \Rightarrow x^2 - 2x - 3 = 0$$
$$\Rightarrow (x-3)(x+1) = 0$$
$$\Rightarrow x = 3 \text{ or } x = -1$$

When $x = 3$, $y = (3)^3 - 3\,(3)^2 - 9\,(3) + 7 = 27 - 27 - 27 + 7 = -20$.

When $x = -1$, $y = (-1)^3 - 3\,(-1)^2 - 9\,(-1) + 7 = -1 - 3 + 9 + 7 = 12$.

Hence, the points at which the tangent is parallel to the x-axis are $(3, -20)$ and

$(-1, 12)$.

8. **Find a point on the curve $y = (x-2)^2$ at which the tangent is parallel to the chord joining the points (2, 0) and (4, 4).**

ANS :

If a tangent is parallel to the chord joining the points (2, 0) and (4, 4), then the slope of the tangent = the slope of the chord.

The slope of the chord is $\frac{4-0}{4-2} = \frac{4}{2} = 2$.

Now, the slope of the tangent to the given curve at a point (x, y) is given by,

$$\frac{dy}{dx} = 2(x-2)$$

Since the slope of the tangent = slope of the chord, we have:

$$2(x-2) = 2$$
$$\Rightarrow x - 2 = 1 \Rightarrow x = 3$$

When $x = 3$, $y = (3-2)^2 = 1$.

Hence, the required point is (3, 1).

9. Find the point on the curve $y = x^3 - 11x + 5$ at which the tangent is $y = x - 11$.

ANS :

The equation of the given curve is $y = x^3 - 11x + 5$.

The equation of the tangent to the given curve is given as $y = x - 11$ (which is of the form $y = mx + c$).

$\therefore$Slope of the tangent = 1

Now, the slope of the tangent to the given curve at the point (x, y) is given by, $\frac{dy}{dx} = 3x^2 - 11$

Then, we have:

$$3x^2 - 11 = 1$$
$$\Rightarrow 3x^2 = 12$$
$$\Rightarrow x^2 = 4$$
$$\Rightarrow x = \pm 2$$

When $x = 2$, $y = (2)^3 - 11(2) + 5 = 8 - 22 + 5 = -9$.

When $x = -2$, $y = (-2)^3 - 11(-2) + 5 = -8 + 22 + 5 = 19$.

Hence, the required points are $(2, -9)$ and $(-2, 19)$.

10. Find the equation of all lines having slope -1 that are tangents to the curve

$$y = \frac{1}{x-1}, x \neq 1.$$

ANS :

The slope of the tangents to the given curve at any point (x, y) is given by,

$$\frac{dy}{dx} = \frac{-1}{(x-1)^2}$$

If the slope of the tangent is -1, then we have:

$$\frac{-1}{(x-1)^2} = -1$$
$$\Rightarrow (x-1)^2 = 1$$
$$\Rightarrow x-1 = \pm 1$$
$$\Rightarrow x = 2,\ 0$$

When $x = 0, y = -1$ and when $x = 2, y = 1$.

Thus, there are two tangents to the given curve having slope -1. These are passing through the points $(0, -1)$ and $(2, 1)$.

$\therefore$The equation of the tangent through $(0, -1)$ is given by,

$$y-(-1) = -1(x-0)$$
$$\Rightarrow y+1 = -x$$
$$\Rightarrow y+x+1 = 0$$

$\therefore$The equation of the tangent through $(2, 1)$ is given by,

$$y - 1 = -1\ (x - 2)$$

$$\Rightarrow y - 1 = -x + 2$$

$$\Rightarrow y + x - 3 = 0$$

Hence, the equations of the required lines are $y + x + 1 = 0$ and $y + x - 3 = 0$.

11. **Find the equation of all lines having slope 2 which are tangents to the curve**

$$y = \frac{1}{x-3}, x \neq 3.$$

The slope of the tangent to the given curve at any point (x, y) is given by,

$$\frac{dy}{dx} = \frac{-1}{(x-3)^2}$$

If the slope of the tangent is 2, then we have:

$$\frac{-1}{(x-3)^2} = 2$$

$$\Rightarrow 2(x-3)^2 = -1$$

$$\Rightarrow (x-3)^2 = \frac{-1}{2}$$

This is not possible since the L.H.S. is positive while the R.H.S. is negative.

Hence, there is no tangent to the given curve having slope 2.

12. **Find the equations of all lines having slope 0 which are tangent to the curve**

$$y = \frac{1}{x^2 - 2x + 3}.$$

The equation of the given curve is $y = \frac{1}{x^2 - 2x + 3}$.

The slope of the tangent to the given curve at any point (x, y) is given by,

$$\frac{dy}{dx} = \frac{-(2x-2)}{(x^2 - 2x + 3)^2} = \frac{-2(x-1)}{(x^2 - 2x + 3)^2}$$

If the slope of the tangent is 0, then we have:

$$\frac{-2(x-1)}{(x^2 - 2x + 3)^2} = 0$$

$$\Rightarrow -2(x-1) = 0$$

$$\Rightarrow x = 1$$

When $x = 1$, $y = \frac{1}{1-2+3} = \frac{1}{2}$.

$\therefore$ The equation of the tangent through $\left(1, \frac{1}{2}\right)$ is given by,

$$y - \frac{1}{2} = 0(x-1)$$

$$\Rightarrow y - \frac{1}{2} = 0$$

$$\Rightarrow y = \frac{1}{2}$$

Hence, the equation of the required line is $y = \frac{1}{2}$.

13. Find points on the curve $\frac{x^2}{9}+\frac{y^2}{16}=1$ at which the tangents are

(i) parallel to x-axis (ii) parallel to y-axis.

ANS :

The equation of the given curve is $\frac{x^2}{9}+\frac{y^2}{16}=1$.

On differentiating both sides with respect to x, we have:

$$\frac{2x}{9}+\frac{2y}{16}\cdot\frac{dy}{dx}=0$$
$$\Rightarrow \frac{dy}{dx}=\frac{-16x}{9y}$$

(i) The tangent is parallel to the x-axis if the slope of the tangent is i.e., $0\frac{-16x}{9y}=0$, which is possible if $x=0$.

Then, $\frac{x^2}{9}+\frac{y^2}{16}=1$ for $x=0$

$\Rightarrow y^2=16 \Rightarrow y=\pm 4$

Hence, the points at which the tangents are parallel to the x-axis are

(0, 4) and (0, − 4).

(ii) The tangent is parallel to the y-axis if the slope of the normal is 0, which gives $\frac{-1}{\left(\frac{-16x}{9y}\right)}=\frac{9y}{16x}=0 \Rightarrow y=0$.

Then, $\frac{x^2}{9}+\frac{y^2}{16}=1$ for $y=0$.

$\Rightarrow x=\pm 3$

Hence, the points at which the tangents are parallel to the y-axis are

(3, 0) and (− 3, 0).

14. Find the equations of the tangent and normal to the given curves at the indicated points:

(i) $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at (0, 5)

(ii) $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at (1, 3)

(iii) $y = x^3$ at (1, 1)

(iv) $y = x^2$ at (0, 0)

(v) $x = \cos t, y = \sin t$ at $t = \frac{\pi}{4}$

ANS :

(i) The equation of the curve is $y = x^4 - 6x^3 + 13x^2 - 10x + 5$.

On differentiating with respect to x, we get:

$$\frac{dy}{dx} = 4x^3 - 18x^2 + 26x - 10$$

$$\left.\frac{dy}{dx}\right]_{(0,\,5)} = -10$$

Thus, the slope of the tangent at (0, 5) is −10. The equation of the tangent is given as:

$y - 5 = -10(x - 0)$

$\Rightarrow y - 5 = -10x$

$\Rightarrow 10x + y = 5$

The slope of the normal at (0, 5) is $\frac{-1}{\text{Slope of the tangent at (0, 5)}} = \frac{1}{10}$.

Therefore, the equation of the normal at (0, 5) is given as:

$$y - 5 = \frac{1}{10}(x - 0)$$
$$\Rightarrow 10y - 50 = x$$
$$\Rightarrow x - 10y + 50 = 0$$

(ii) The equation of the curve is $y = x^4 - 6x^3 + 13x^2 - 10x + 5$.

On differentiating with respect to x, we get:

$$\frac{dy}{dx} = 4x^3 - 18x^2 + 26x - 10$$
$$\left.\frac{dy}{dx}\right]_{(1,\,3)} = 4 - 18 + 26 - 10 = 2$$

Thus, the slope of the tangent at (1, 3) is 2. The equation of the tangent is given as:

$$y - 3 = 2(x - 1)$$
$$\Rightarrow y - 3 = 2x - 2$$
$$\Rightarrow y = 2x + 1$$

The slope of the normal at (1, 3) is $\dfrac{-1}{\text{Slope of the tangent at (1, 3)}} = \dfrac{-1}{2}$.

Therefore, the equation of the normal at (1, 3) is given as:

$$y - 3 = -\frac{1}{2}(x - 1)$$
$$\Rightarrow 2y - 6 = -x + 1$$
$$\Rightarrow x + 2y - 7 = 0$$

(iii) The equation of the curve is $y = x^3$.

On differentiating with respect to x, we get:

$$\frac{dy}{dx} = 3x^2$$
$$\left.\frac{dy}{dx}\right]_{(1,\ 1)} = 3(1)^2 = 3$$

Thus, the slope of the tangent at (1, 1) is 3 and the equation of the tangent is given as:

$$y - 1 = 3(x - 1)$$
$$\Rightarrow y = 3x - 2$$

The slope of the normal at (1, 1) is $\frac{-1}{\text{Slope of the tangent at } (1,\ 1)} = \frac{-1}{3}$.

Therefore, the equation of the normal at (1, 1) is given as:

$$y - 1 = \frac{-1}{3}(x - 1)$$
$$\Rightarrow 3y - 3 = -x + 1$$
$$\Rightarrow x + 3y - 4 = 0$$

(iv) The equation of the curve is $y = x^2$.

On differentiating with respect to x, we get:

$$\frac{dy}{dx} = 2x$$
$$\left.\frac{dy}{dx}\right]_{(0,\ 0)} = 0$$

Thus, the slope of the tangent at (0, 0) is 0 and the equation of the tangent is given as:

$$y - 0 = 0\ (x - 0)$$

$$\Rightarrow y = 0$$

The slope of the normal at (0, 0) is $\frac{-1}{\text{Slope of the tangent at } (0,\ 0)} = -\frac{1}{0}$, which is not defined.

Therefore, the equation of the normal at $(x_0, y_0) = (0, 0)$ is given by

$$x = x_0 = 0.$$

(v) The equation of the curve is $x = \cos t, y = \sin t$.

$$x = \cos t \text{ and } y = \sin t$$
$$\therefore \frac{dx}{dt} = -\sin t,\ \frac{dy}{dt} = \cos t$$
$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{\cos t}{-\sin t} = -\cot t$$
$$\left.\frac{dy}{dx}\right]_{t=\frac{\pi}{4}} = -\cot t = -1$$

$\therefore$The slope of the tangent at $t = \frac{\pi}{4}$ is -1.

When $t = \frac{\pi}{4}$, $x = \frac{1}{\sqrt{2}}$ and $y = \frac{1}{\sqrt{2}}$.

Thus, the equation of the tangent to the given curve at $t = \frac{\pi}{4}$ i.e., at $\left[\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right]$ is

$$y - \frac{1}{\sqrt{2}} = -1\left(x - \frac{1}{\sqrt{2}}\right).$$
$$\Rightarrow x + y - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = 0$$
$$\Rightarrow x + y - \sqrt{2} = 0$$

The slope of the normal at $t = \frac{\pi}{4}$ is $\frac{-1}{\text{Slope of the tangent at } t = \frac{\pi}{4}} = 1.$

Therefore, the equation of the normal to the given curve at $t = \frac{\pi}{4}$ i.e., at $\left[\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right]$ is

$$y - \frac{1}{\sqrt{2}} = 1\left(x - \frac{1}{\sqrt{2}}\right).$$
$$\Rightarrow x = y$$

15. Find the equation of the tangent line to the curve $y = x^2 - 2x + 7$ which is

(a) parallel to the line $2x - y + 9 = 0$

(b) perpendicular to the line $5y - 15x = 13$.

ANS :

The equation of the given curve is $y = x^2 - 2x + 7$.

On differentiating with respect to x, we get:

$$\frac{dy}{dx} = 2x - 2$$

(a) The equation of the line is $2x - y + 9 = 0$.

$2x - y + 9 = 0 \Rightarrow y = 2x + 9$

This is of the form $y = mx + c$.

$\therefore$Slope of the line $= 2$

If a tangent is parallel to the line $2x - y + 9 = 0$, then the slope of the tangent is equal to the slope of the line.

Therefore, we have:

$2 = 2x - 2$

$\Rightarrow 2x = 4$
$\Rightarrow x = 2$

Now, $x = 2$

$\Rightarrow y = 4 - 4 + 7 = 7$

Thus, the equation of the tangent passing through (2, 7) is given by,

$y - 7 = 2(x - 2)$
$\Rightarrow y - 2x - 3 = 0$

Hence, the equation of the tangent line to the given curve (which is parallel to line $2x - y + 9 = 0$) is $y - 2x - 3 = 0$.

(b) The equation of the line is $5y - 15x = 13$.

$5y - 15x = 13 \Rightarrow \; y = 3x + \frac{13}{5}$

This is of the form $y = mx + c$.

$\therefore$Slope of the line $= 3$

If a tangent is perpendicular to the line $5y - 15x = 13$, then the slope of the tangent is $\frac{-1}{\text{slope of the line}} = \frac{-1}{3}$.

$$\Rightarrow 2x - 2 = \frac{-1}{3}$$
$$\Rightarrow 2x = \frac{-1}{3} + 2$$
$$\Rightarrow 2x = \frac{5}{3}$$
$$\Rightarrow x = \frac{5}{6}$$

Now, $x = \frac{5}{6}$

$$\Rightarrow y = \frac{25}{36} - \frac{10}{6} + 7 = \frac{25 - 60 + 252}{36} = \frac{217}{36}$$

Thus, the equation of the tangent passing through $\left(\frac{5}{6}, \; \frac{217}{36}\right)$ is given by,

$$y - \frac{217}{36} = -\frac{1}{3}\left(x - \frac{5}{6}\right)$$
$$\Rightarrow \frac{36y - 217}{36} = \frac{-1}{18}(6x - 5)$$
$$\Rightarrow 36y - 217 = -2(6x - 5)$$
$$\Rightarrow 36y - 217 = -12x + 10$$
$$\Rightarrow 36y + 12x - 227 = 0$$

Hence, the equation of the tangent line to the given curve (which is perpendicular to line $5y - 15x = 13$) is $36y + 12x - 227 = 0$.

16. **Show that the tangents to the curve $y = 7x^3 + 11$ at the points where $x = 2$ and $x = -2$ are parallel.**

ANS :

The equation of the given curve is $y = 7x^3 + 11$.

$$\therefore \frac{dy}{dx} = 21x^2$$

The slope of the tangent to a curve at (x_0, y_0) is $\left.\frac{dy}{dx}\right]_{(x_0, y_0)}$.

Therefore, the slope of the tangent at the point where $x = 2$ is given by,

$$\left.\frac{dy}{dx}\right]_{x=-2} = 21(2)^2 = 84$$

It is observed that the slopes of the tangents at the points where $x = 2$ and $x = -2$ are equal.

Hence, the two tangents are parallel.

17. **Find the points on the curve $y = x^3$ at which the slope of the tangent is equal to the y-coordinate of the point.**

The equation of the given curve is $y = x^3$.

$$\therefore \frac{dy}{dx} = 3x^2$$

The slope of the tangent at the point (x, y) is given by,

$$\left.\frac{dy}{dx}\right]_{(x,y)} = 3x^2$$

When the slope of the tangent is equal to the y-coordinate of the point, then $y = 3x^2$.

Also, we have $y = x^3$.

$$\therefore 3x^2 = x^3$$

$$\Rightarrow x^2(x - 3) = 0$$

$$\Rightarrow x = 0, x = 3$$

When $x = 0$, then $y = 0$ and when $x = 3$, then $y = 3(3)^2 = 27$.

Hence, the required points are (0, 0) and (3, 27).

18. For the curve $y = 4x^3 - 2x^5$, find all the points at which the tangent passes through the origin.

ANS :

The equation of the given curve is $y = 4x^3 - 2x^5$.

$$\therefore \frac{dy}{dx} = 12x^2 - 10x^4$$

Therefore, the slope of the tangent at a point (x, y) is $12x^2 - 10x^4$.

The equation of the tangent at (x, y) is given by,

$$Y - y = \left(12x^2 - 10x^4\right)(X - x) \qquad ...(1)$$

When the tangent passes through the origin (0, 0), then $X = Y = 0$.

Therefore, equation (1) reduces to:

$$-y = \left(12x^2 - 10x^4\right)(-x)$$
$$y = 12x^3 - 10x^5$$

Also, we have $y = 4x^3 - 2x^5$.

$$\therefore 12x^3 - 10x^5 = 4x^3 - 2x^5$$
$$\Rightarrow 8x^5 - 8x^3 = 0$$
$$\Rightarrow x^5 - x^3 = 0$$
$$\Rightarrow x^3\left(x^2 - 1\right) = 0$$
$$\Rightarrow x = 0, \ \pm 1$$

When $x = 0$, $y = 4(0)^3 - 2(0)^5 = 0$.

When $x = 1$, $y = 4\,(1)^3 - 2\,(1)^5 = 2$.

When $x = -1$, $y = 4\,(-1)^3 - 2\,(-1)^5 = -2$.

Hence, the required points are (0, 0), (1, 2), and (−1, −2).

19. Find the points on the curve $x^2 + y^2 - 2x - 3 = 0$ at which the tangents are parallel to the x-axis.

ANS :

The equation of the given curve is $x^2 + y^2 - 2x - 3 = 0$.

On differentiating with respect to x, we have:

$$2x + 2y\frac{dy}{dx} - 2 = 0$$
$$\Rightarrow y\frac{dy}{dx} = 1 - x$$
$$\Rightarrow \frac{dy}{dx} = \frac{1-x}{y}$$

Now, the tangents are parallel to the x-axis if the slope of the tangent is 0.

$$\therefore \frac{1-x}{y} = 0 \Rightarrow 1 - x = 0 \Rightarrow x = 1$$

But, $x^2 + y^2 - 2x - 3 = 0$ for $x = 1$.

$$\Rightarrow y^2 = 4 \Rightarrow y = \pm 2$$

Hence, the points at which the tangents are parallel to the x-axis are $(1, 2)$ and $(1, -2)$.

20. Find the equation of the normal at the point (am^2, am^3) for the curve $ay^2 = x^3$.

ANS :

The equation of the given curve is $ay^2 = x^3$.

On differentiating with respect to x, we have:

$$2ay\frac{dy}{dx} = 3x^2$$
$$\Rightarrow \frac{dy}{dx} = \frac{3x^2}{2ay}$$

The slope of a tangent to the curve at (x_0, y_0) is $\left.\frac{dy}{dx}\right]_{(x_0, y_0)}$.

$\Rightarrow$ The slope of the tangent to the given curve at (am^2, am^3) is

$$\left.\frac{dy}{dx}\right]_{(am^2,\ am^3)} = \frac{3(am^2)^2}{2a(am^3)} = \frac{3a^2m^4}{2a^2m^3} = \frac{3m}{2}.$$

$\therefore$ Slope of normal at (am^2, am^3)

$$= \frac{-1}{\text{slope of the tangent at } (am^2,\ am^3)} = \frac{-2}{3m}$$

Hence, the equation of the normal at (am^2, am^3) is given by,

$$y - am^3 = \frac{-2}{3m}(x - am^2)$$

$$\Rightarrow 3my - 3am^4 = -2x + 2am^2$$
$$\Rightarrow 2x + 3my - am^2(2 + 3m^2) = 0$$

21. Find the equation of the normals to the curve $y = x^3 + 2x + 6$ which are parallel to the line $x + 14y + 4 = 0$.

ANS :

The equation of the given curve is $y = x^3 + 2x + 6$.

The slope of the tangent to the given curve at any point (x, y) is given by,

$$\frac{dy}{dx} = 3x^2 + 2$$

$\therefore$ Slope of the normal to the given curve at any point (x, y)

$$= \frac{-1}{\text{Slope of the tangent at the point } (x, y)}$$
$$= \frac{-1}{3x^2 + 2}$$

The equation of the given line is $x + 14y + 4 = 0$.

$x + 14y + 4 = 0 \Rightarrow y = -\frac{1}{14}x - \frac{4}{14}$ (which is of the form $y = mx + c$)

$\therefore$Slope of the given line $= \frac{-1}{14}$

If the normal is parallel to the line, then we must have the slope of the normal being equal to the slope of the line.

$$\therefore \frac{-1}{3x^2 + 2} = \frac{-1}{14}$$
$$\Rightarrow 3x^2 + 2 = 14$$
$$\Rightarrow 3x^2 = 12$$
$$\Rightarrow x^2 = 4$$
$$\Rightarrow x = \pm 2$$

When $x = 2, y = 8 + 4 + 6 = 18$.

When $x = -2, y = -8 - 4 + 6 = -6$.

Therefore, there are two normals to the given curve with slope $\frac{-1}{14}$ and passing through the points (2, 18) and (−2, −6).

Thus, the equation of the normal through (2, 18) is given by,

$$y - 18 = \frac{-1}{14}(x - 2)$$
$$\Rightarrow 14y - 252 = -x + 2$$
$$\Rightarrow x + 14y - 254 = 0$$

And, the equation of the normal through (−2, −6) is given by,

$$y - (-6) = \frac{-1}{14}[x - (-2)]$$
$$\Rightarrow y + 6 = \frac{-1}{14}(x + 2)$$
$$\Rightarrow 14y + 84 = -x - 2$$
$$\Rightarrow x + 14y + 86 = 0$$

Hence, the equations of the normals to the given curve (which are parallel to the given line) are $x + 14y - 254 = 0$ and $x + 14y + 86 = 0$.

22. Find the equations of the tangent and normal to the parabola $y^2 = 4ax$ at the point $(at^2,\ 2at)$.

ANS :

The equation of the given parabola is $y^2 = 4ax$.

On differentiating $y^2 = 4ax$ with respect to x, we have:

$$2y\frac{dy}{dx} = 4a$$
$$\Rightarrow \frac{dy}{dx} = \frac{2a}{y}$$

$\therefore$The slope of the tangent at $(at^2,\ 2at)$ is $\left.\frac{dy}{dx}\right]_{(at^2,\ 2at)} = \frac{2a}{2at} = \frac{1}{t}$.

Then, the equation of the tangent at $(at^2,\ 2at)$ is given by,

$$y - 2at = \frac{1}{t}\left(x - at^2\right)$$

$$\Rightarrow ty - 2at^2 = x - at^2$$
$$\Rightarrow ty = x + at^2$$

Now, the slope of the normal at $(at^2,\ 2at)$ is given by,

$$\frac{-1}{\text{Slope of the tangent at } \left(at^2, 2at\right)} = -t$$

Thus, the equation of the normal at $(at^2, 2at)$ is given as:

$$y - 2at = -t\left(x - at^2\right)$$
$$\Rightarrow y - 2at = -tx + at^3$$
$$\Rightarrow y = -tx + 2at + at^3$$

23. Prove that the curves $x = y^2$ and $xy = k$ cut at right angles* if $8k^2 = 1$.

ANS :

The equations of the given curves are given as $x = y^2$ and $xy = k$.

Putting $x = y^2$ in $xy = k$, we get:

$$y^3 = k \Rightarrow y = k^{\frac{1}{3}}$$
$$\therefore x = k^{\frac{2}{3}}$$

Thus, the point of intersection of the given curves is $\left(k^{\frac{2}{3}}, k^{\frac{1}{3}}\right)$.

Differentiating $x = y^2$ with respect to x, we have:

$$1 = 2y\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{2y}$$

Therefore, the slope of the tangent to the curve $x = y^2$ at $\left(k^{\frac{2}{3}}, k^{\frac{1}{3}}\right)$ is $\left.\frac{dy}{dx}\right]_{\left(k^{\frac{2}{3}}, k^{\frac{1}{3}}\right)} = \frac{1}{2k^{\frac{1}{3}}}$.

On differentiating $xy = k$ with respect to x, we have:

$$x\frac{dy}{dx} + y = 0 \Rightarrow \frac{dy}{dx} = \frac{-y}{x}$$

$\therefore$ Slope of the tangent to the curve $xy = k$ at $\left(k^{\frac{2}{3}}, k^{\frac{1}{3}}\right)$ is given by,

$$\left.\frac{dy}{dx}\right]_{\left(k^{\frac{2}{3}}, k^{\frac{1}{3}}\right)} = \left.\frac{-y}{x}\right]_{\left(k^{\frac{2}{3}}, k^{\frac{1}{3}}\right)} = -\frac{k^{\frac{1}{3}}}{k^{\frac{2}{3}}} = \frac{-1}{k^{\frac{1}{3}}}$$

We know that two curves intersect at right angles if the tangents to the curves at the point of intersection i.e., at $\left(k^{\frac{2}{3}}, k^{\frac{1}{3}}\right)$ are perpendicular to each other.

This implies that we should have the product of the tangents as -1.

Thus, the given two curves cut at right angles if the product of the slopes of their respective tangents at $\left(k^{\frac{2}{3}}, k^{\frac{1}{3}}\right)$ is -1.

i.e., $\left(\frac{1}{2k^{\frac{1}{3}}}\right)\left(\frac{-1}{k^{\frac{1}{3}}}\right) = -1$

$$\Rightarrow 2k^{\frac{2}{3}} = 1$$
$$\Rightarrow \left(2k^{\frac{2}{3}}\right)^3 = (1)^3$$
$$\Rightarrow 8k^2 = 1$$

Hence, the given two curves cut at right angels if $8k^2 = 1$.

24. Find the equations of the tangent and normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point (x_0, y_0).

ANS :

Differentiating $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with respect to x, we have:

$$\frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0$$
$$\Rightarrow \frac{2y}{b^2}\frac{dy}{dx} = \frac{2x}{a^2}$$
$$\Rightarrow \frac{dy}{dx} = \frac{b^2 x}{a^2 y}$$

Therefore, the slope of the tangent at (x_0, y_0) is $\left.\frac{dy}{dx}\right]_{(x_0, y_0)} = \frac{b^2 x_0}{a^2 y_0}$.

Then, the equation of the tangent at (x_0, y_0) is given by,

$$y - y_0 = \frac{b^2 x_0}{a^2 y_0}(x - x_0)$$
$$\Rightarrow a^2 y y_0 - a^2 y_0^2 = b^2 x x_0 - b^2 x_0^2$$
$$\Rightarrow b^2 x x_0 - a^2 y y_0 - b^2 x_0^2 + a^2 y_0^2 = 0$$
$$\Rightarrow \frac{x x_0}{a^2} - \frac{y y_0}{b^2} - \left(\frac{x_0^2}{a^2} - \frac{y_0^2}{b^2}\right) = 0 \quad \left[\text{On dividing both sides by } a^2 b^2\right]$$
$$\Rightarrow \frac{x x_0}{a^2} - \frac{y y_0}{b^2} - 1 = 0 \quad \left[(x_0, y_0) \text{ lies on the hyperbola } \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\right]$$
$$\Rightarrow \frac{x x_0}{a^2} - \frac{y y_0}{b^2} = 1$$

Now, the slope of the normal at (x_0, y_0) is given by,

$$\frac{-1}{\text{Slope of the tangent at } (x_0, y_0)} = \frac{-a^2 y_0}{b^2 x_0}$$

Hence, the equation of the normal at (x_0, y_0) is given by,

$$y - y_0 = \frac{-a^2 y_0}{b^2 x_0}(x - x_0)$$
$$\Rightarrow \frac{y - y_0}{a^2 y_0} = \frac{-(x - x_0)}{b^2 x_0}$$
$$\Rightarrow \frac{y - y_0}{a^2 y_0} + \frac{(x - x_0)}{b^2 x_0} = 0$$

25. Find the equation of the tangent to the curve $y=\sqrt{3x-2}$ which is parallel to the line $4x-2y+5=0$.

ANS :

The equation of the given curve is $y=\sqrt{3x-2}$.

The slope of the tangent to the given curve at any point (x, y) is given by,

$$\frac{dy}{dx}=\frac{3}{2\sqrt{3x-2}}$$

The equation of the given line is $4x-2y+5=0$.

$4x-2y+5=0 \Rightarrow y=2x+\frac{5}{2}$ (which is of the form $y=mx+c$)

$\therefore$Slope of the line $=2$

Now, the tangent to the given curve is parallel to the line $4x-2y-5=0$ if the slope of the tangent is equal to the slope of the line.

$$\frac{3}{2\sqrt{3x-2}}=2$$
$$\Rightarrow \sqrt{3x-2}=\frac{3}{4}$$
$$\Rightarrow 3x-2=\frac{9}{16}$$
$$\Rightarrow 3x=\frac{9}{16}+2=\frac{41}{16}$$
$$\Rightarrow x=\frac{41}{48}$$

When $x=\frac{41}{48}$, $y=\sqrt{3\left(\frac{41}{48}\right)-2}=\sqrt{\frac{41}{16}-2}=\sqrt{\frac{41-32}{16}}=\sqrt{\frac{9}{16}}=\frac{3}{4}$.

$\therefore$Equation of the tangent passing through the point $\left(\frac{41}{48}, \frac{3}{4}\right)$ is given by,

$$y-\frac{3}{4}=2\left(x-\frac{41}{48}\right)$$
$$\Rightarrow \frac{4y-3}{4}=2\left(\frac{48x-41}{48}\right)$$
$$\Rightarrow 4y-3=\frac{48x-41}{6}$$
$$\Rightarrow 24y-18=48x-41$$
$$\Rightarrow 48x-24y=23$$

Hence, the equation of the required tangent is $48x-24y=23$.

26. The slope of the normal to the curve $y = 2x^2 + 3\sin x$ at $x = 0$ is

(A) 3 (B) $\frac{1}{3}$ (C) -3 (D) $-\frac{1}{3}$

ANS :

The equation of the given curve is $y = 2x^2 + 3\sin x$.

Slope of the tangent to the given curve at $x = 0$ is given by,

$$\left.\frac{dy}{dx}\right]_{x=0} = 4x + 3\cos x\Big]_{x=0} = 0 + 3\cos 0 = 3$$

Hence, the slope of the normal to the given curve at $x = 0$ is

$$\frac{-1}{\text{Slope of the tangent at } x = 0} = \frac{-1}{3}.$$

The correct answer is D.

27. The line $y = x + 1$ is a tangent to the curve $y^2 = 4x$ at the point

(A) (1, 2) (B) (2, 1) (C) (1, – 2) (D) (– 1, 2)

ANS :

The equation of the given curve is $y^2 = 4x$.

Differentiating with respect to x, we have:

$$2y\frac{dy}{dx} = 4 \Rightarrow \frac{dy}{dx} = \frac{2}{y}$$

Therefore, the slope of the tangent to the given curve at any point (x, y) is given by,

$$\frac{dy}{dx} = \frac{2}{y}$$

The given line is $y = x + 1$ (which is of the form $y = mx + c$)

$\therefore$ Slope of the line = 1

The line $y = x + 1$ is a tangent to the given curve if the slope of the line is equal to the slope of the tangent. Also, the line must intersect the curve.

Thus, we must have:

$$\frac{2}{y} = 1$$
$$\Rightarrow y = 2$$
Now, $y = x + 1 \Rightarrow x = y - 1 \Rightarrow x = 2 - 1 = 1$

Hence, the line $y = x + 1$ is a tangent to the given curve at the point (1, 2).

The correct answer is A.