CLASS XII APPLICATION OF DERIVATIVES CHAPTER 6

EX. 6.1 SOLUTIONS

1. Find the rate of change of the area of a circle with respect to its radius r when
   (a) $r = 3$ cm (b) $r = 4$ cm

ANS :

The area of a circle (A) with radius (r) is given by,

$A = \pi r^2$

Now, the rate of change of the area with respect to its radius is given by, $\frac{dA}{dr} = \frac{d}{dr}\left(\pi r^2\right) = 2\pi r$

1. When $r = 3$ cm,

$$\frac{dA}{dr} = 2\pi(3) = 6\pi$$

Hence, the area of the circle is changing at the rate of 6π cm^2/s when its radius is 3 cm.

2. When $r = 4$ cm,

$$\frac{dA}{dr} = 2\pi(4) = 8\pi$$

Hence, the area of the circle is changing at the rate of 8π cm^2/s when its radius is 4 cm.

2. The volume of a cube is increasing at the rate of 8 cm^3/s. How fast is the surface area increasing when the length of an edge is 12 cm?

ANS :

Let x be the length of a side, V be the volume, and s be the surface area of the cube.

Then, $V = x^3$ and $S = 6x^2$ where x is a function of time t.

It is given that $\frac{dV}{dt} = 8 \text{ cm}^3/\text{s}$.

Then, by using the chain rule, we have:

$$\therefore 8 = \frac{dV}{dt} = \frac{d}{dt}\left(x^3\right) = \frac{d}{dx}\left(x^3\right)\cdot\frac{dx}{dt} = 3x^2.\frac{dx}{dt}$$

$$\Rightarrow \frac{dx}{dt} = \frac{8}{3x^2} \qquad (1)$$

$$\text{Now, } \frac{dS}{dt} = \frac{d}{dt}\left(6x^2\right) = \frac{d}{dx}\left(6x^2\right)\cdot\frac{dx}{dt} \qquad \text{[By chain rule]}$$

$$= 12x.\frac{dx}{dt} = 12x\cdot\left(\frac{8}{3x^2}\right) = \frac{32}{x}$$

Thus, when $x = 12$ cm, $\frac{dS}{dt} = \frac{32}{12} \text{ cm}^2/\text{s} = \frac{8}{3} \text{ cm}^2/\text{s}.$

Hence, if the length of the edge of the cube is 12 cm, then the surface area is increasing at the rate of $\frac{8}{3}$ cm^2/s.

3. The radius of a circle is increasing uniformly at the rate of 3 cm/s. Find the rate at which the area of the circle is increasing when the radius is 10 cm.

ANS:

The area of a circle (*A*) with radius (*r*) is given by,

$$A = \pi r^2$$

Now, the rate of change of area (*A*) with respect to time (*t*) is given by,

$$\frac{dA}{dt} = \frac{d}{dt}\left(\pi r^2\right)\cdot\frac{dr}{dt} = 2\pi r\frac{dr}{dt} \qquad \text{[By chain rule]}$$

It is given that,

$$\frac{dr}{dt} = 3 \text{ cm/s}$$

$$\therefore \frac{dA}{dt} = 2\pi r(3) = 6\pi r$$

Thus, when $r = 10$ cm,

$$\frac{dA}{dt} = 6\pi(10) = 60\pi \text{ cm}^2/\text{s}$$

Hence, the rate at which the area of the circle is increasing when the radius is 10 cm is 60π cm^2/s.

4. An edge of a variable cube is increasing at the rate of 3 cm/s. How fast is the volume of the cube increasing when the edge is 10 cm long?

ANS:

Let x be the length of a side and V be the volume of the cube. Then,

$V = x^3.$

$$\therefore \frac{dV}{dt} = 3x^2 \cdot \frac{dx}{dt} \text{ (By chain rule)}$$

It is given that,

$$\frac{dx}{dt} = 3 \text{ cm/s}$$

$$\therefore \frac{dV}{dt} = 3x^2(3) = 9x^2$$

Thus, when $x = 10$ cm,

$$\frac{dV}{dt} = 9(10)^2 = 900 \text{ cm}^3/\text{s}$$

Hence, the volume of the cube is increasing at the rate of 900 cm^3/s when the edge is 10 cm long.

5. A stone is dropped into a quiet lake and waves move in circles at the speed of 5 cm/s. At the instant when the radius of the circular wave is 8 cm, how fast is the enclosed area increasing?

ANS:

The area of a circle (A) with radius (r) is given by $A = \pi r^2$.

Therefore, the rate of change of area (A) with respect to time (t) is given by,

$$\frac{dA}{dt} = \frac{d}{dt}\left(\pi r^2\right) = \frac{d}{dr}\left(\pi r^2\right)\frac{dr}{dt} = 2\pi r\frac{dr}{dt}$$ [By chain rule]

It is given that $\frac{dr}{dt} = 5 \text{ cm/s}$.

Thus, when $r = 8$ cm,

$$\frac{dA}{dt} = 2\pi(8)(5) = 80\pi$$

Hence, when the radius of the circular wave is 8 cm, the enclosed area is increasing at the rate of 80π cm^2/s.

6. The radius of a circle is increasing at the rate of 0.7 cm/s. What is the rate of increase of its circumference?

ANS:

The circumference of a circle (C) with radius (r) is given by

$C = 2\pi r$.

Therefore, the rate of change of circumference (C) with respect to time (t) is given by,

$$\frac{dC}{dt} = \frac{dC}{dr}\cdot\frac{dr}{dt}$$ (By chain rule)

$$= \frac{d}{dr}(2\pi r)\,\frac{dr}{dt}$$

$$= 2\pi\cdot\frac{dr}{dt}$$

It is given that $\frac{dr}{dt} = 0.7 \text{ cm/s}$.

Hence, the rate of increase of the circumference is $2\pi(0.7) = 1.4\pi$ cm/s.

7. The length x of a rectangle is decreasing at the rate of 5 cm/minute and the width y is increasing at the rate of 4 cm/minute. When $x = 8$cm and $y = 6$cm, find the rates of change of (a) the perimeter, and (b) the area of the rectangle.

ANS:

Since the length (x) is decreasing at the rate of 5 cm/minute and the width (y) is increasing at the rate of 4 cm/minute, we have:

$$\frac{dx}{dt} = -5 \text{ cm/min and } \frac{dy}{dt} = 4 \text{ cm/min}$$

(a) The perimeter (P) of a rectangle is given by,

$$P = 2(x + y)$$

$$\therefore \frac{dP}{dt} = 2\left(\frac{dx}{dt} + \frac{dy}{dt}\right) = 2(-5+4) = -2 \text{ cm/min}$$

Hence, the perimeter is decreasing at the rate of 2 cm/min.

(b) The area (A) of a rectangle is given by,

$$A = x \times y$$

$$\therefore \frac{dA}{dt} = \frac{dx}{dt} \cdot y + x \cdot \frac{dy}{dt} = -5y + 4x$$

When $x = 8$ cm and $y = 6$ cm, $\frac{dA}{dt} = (-5\times6+4\times8) \text{ cm}^2/\text{min} = 2 \text{ cm}^2/\text{min}$

Hence, the area of the rectangle is increasing at the rate of 2 cm^2/min.

8. A balloon, which always remains spherical on inflation, is being inflated by pumping in 900 cubic centimetres of gas per second. Find the rate at which the radius of the balloon increases when the radius is 15 cm.

ANS:

The volume of a sphere (V) with radius (r) is given by,

$$V = \frac{4}{3}\pi r^3$$

$\therefore$Rate of change of volume (V) with respect to time (t) is given by,

$$\frac{dV}{dt} = \frac{dV}{dr}\cdot\frac{dr}{dt} \text{ [By chain rule]}$$

$$= \frac{d}{dr}\left(\frac{4}{3}\pi r^3\right)\cdot\frac{dr}{dt}$$

$$= 4\pi r^2 \cdot \frac{dr}{dt}$$

It is given that $\frac{dV}{dt} = 900 \text{ cm}^3/\text{s}$.

$$\therefore 900 = 4\pi r^2 \cdot \frac{dr}{dt}$$

$$\Rightarrow \frac{dr}{dt} = \frac{900}{4\pi r^2} = \frac{225}{\pi r^2}$$

Therefore, when radius = 15 cm,

$$\frac{dr}{dt} = \frac{225}{\pi(15)^2} = \frac{1}{\pi}$$

Hence, the rate at which the radius of the balloon increases when the radius is 15 cm is $\frac{1}{\pi}$ cm/s.

9. A balloon, which always remains spherical has a variable radius. Find the rate at which its volume is increasing with the radius when the later is 10 cm.

ANS:

The volume of a sphere (V) with radius (r) is given by $V = \frac{4}{3}\pi r^3$.

Rate of change of volume (V) with respect to its radius (r) is given by,

$$\frac{dV}{dr} = \frac{d}{dr}\left(\frac{4}{3}\pi r^3\right) = \frac{4}{3}\pi\left(3r^2\right) = 4\pi r^2$$

Therefore, when radius = 10 cm,

$$\frac{dV}{dr} = 4\pi(10)^2 = 400\pi$$

Hence, the volume of the balloon is increasing at the rate of 400π cm^3/s.

10. A ladder 5 m long is leaning against a wall. The bottom of the ladder is pulled along the ground, away from the wall, at the rate of 2cm/s. How fast is its height on the wall decreasing when the foot of the ladder is 4 m away from the wall ?

ANS:

Let y m be the height of the wall at which the ladder touches. Also, let the foot of the ladder be x maway from the wall.

Then, by Pythagoras theorem, we have:

$x^2 + y^2 = 25$ [Length of the ladder = 5 m]

$$\Rightarrow y = \sqrt{25 - x^2}$$

Then, the rate of change of height (y) with respect to time (t) is given by,

$$\frac{dy}{dt} = \frac{-x}{\sqrt{25 - x^2}} \cdot \frac{dx}{dt}$$

It is given that $\frac{dx}{dt} = 2$ cm/s.

$$\therefore \frac{dy}{dt} = \frac{-2x}{\sqrt{25 - x^2}}$$

Now, when $x = 4$ m, we have:

$$\frac{dy}{dt} = \frac{-2 \times 4}{\sqrt{25 - 4^2}} = -\frac{8}{3}$$

Hence, the height of the ladder on the wall is decreasing at the rate of $\frac{8}{3}$ cm/s.

11. A particle moves along the curve $6y = x^3 + 2$. Find the points on the curve at which the y-coordinate is changing 8 times as fast as the x-coordinate.

ANS:

The equation of the curve is given as:

$$6y = x^3 + 2$$

The rate of change of the position of the particle with respect to time (*t*) is given by,

$$6\frac{dy}{dt} = 3x^2\frac{dx}{dt} + 0$$
$$\Rightarrow 2\frac{dy}{dt} = x^2\frac{dx}{dt}$$

When the *y*-coordinate of the particle changes 8 times as fast as the

x-coordinate i.e., $\left(\frac{dy}{dt} = 8\frac{dx}{dt}\right)$, we have:

$$2\left(8\frac{dx}{dt}\right) = x^2\frac{dx}{dt}$$
$$\Rightarrow 16\frac{dx}{dt} = x^2\frac{dx}{dt}$$
$$\Rightarrow \left(x^2 - 16\right)\frac{dx}{dt} = 0$$
$$\Rightarrow x^2 = 16$$
$$\Rightarrow x = \pm 4$$

When $x = 4, y = \frac{4^3 + 2}{6} = \frac{66}{6} = 11$.

When $x = -4, y = \frac{(-4)^3 + 2}{6} = -\frac{62}{6} = -\frac{31}{3}$.

Hence, the points required on the curve are (4, 11) and $\left(-4, \frac{-31}{3}\right)$.

12. The radius of an air bubble is increasing at the rate of $\frac{1}{2}$ cm/s. At what rate is the volume of the bubble increasing when the radius is 1 cm?

ANS:

The air bubble is in the shape of a sphere.

Now, the volume of an air bubble (V) with radius (r) is given by,

$$V = \frac{4}{3}\pi r^3$$

The rate of change of volume (V) with respect to time (t) is given by,

$$\frac{dV}{dt} = \frac{4}{3}\pi \frac{d}{dr}\left(r^3\right)\cdot\frac{dr}{dt} \qquad \text{[By chain rule]}$$

$$= \frac{4}{3}\pi\left(3r^2\right)\frac{dr}{dt}$$

$$= 4\pi r^2 \frac{dr}{dt}$$

It is given that $\frac{dr}{dt} = \frac{1}{2}$ cm/s.

Therefore, when $r = 1$ cm,

$$\frac{dV}{dt} = 4\pi(1)^2\left(\frac{1}{2}\right) = 2\pi \text{ cm}^3/\text{s}$$

Hence, the rate at which the volume of the bubble increases is 2π cm^3/s.

13. A balloon, which always remains spherical, has a variable diameter $\frac{3}{2}(2x+1)$. Find the rate of change of its volume with respect to x.

ANS:

The volume of a sphere (V) with radius (r) is given by,

$$V=\frac{4}{3}\pi r^3$$

It is given that:

$$\text{Diameter}=\frac{3}{2}(2x+1)$$

$$\Rightarrow r=\frac{3}{4}(2x+1)$$

$$\therefore V=\frac{4}{3}\pi\left(\frac{3}{4}\right)^3(2x+1)^3=\frac{9}{16}\pi(2x+1)^3$$

Hence, the rate of change of volume with respect to x is as

$$\frac{dV}{dx}=\frac{9}{16}\pi\frac{d}{dx}(2x+1)^3=\frac{9}{16}\pi\times 3(2x+1)^2\times 2=\frac{27}{8}\pi(2x+1)^2.$$

14. Sand is pouring from a pipe at the rate of 12 cm³/s. The falling sand forms a cone on the ground in such a way that the height of the cone is always one-sixth of the radius of the base. How fast is the height of the sand cone increasing when the height is 4 cm?

ANS:

The volume of a cone (V) with radius (r) and height (h) is given by,

$$V = \frac{1}{3}\pi r^2 h$$

It is given that,

$$h = \frac{1}{6}r \Rightarrow r = 6h$$

$$\therefore V = \frac{1}{3}\pi(6h)^2 h = 12\pi h^3$$

The rate of change of volume with respect to time (t) is given by,

$$\frac{dV}{dt} = 12\pi \frac{d}{dh}\left(h^3\right)\cdot\frac{dh}{dt} \text{ [By chain rule]}$$

$$= 12\pi\left(3h^2\right)\frac{dh}{dt}$$
$$= 36\pi h^2 \frac{dh}{dt}$$

It is also given that $\frac{dV}{dt} = 12 \text{ cm}^3/\text{s}$.

Therefore, when $h = 4$ cm, we have:

$$12 = 36\pi(4)^2 \frac{dh}{dt}$$
$$\Rightarrow \frac{dh}{dt} = \frac{12}{36\pi(16)} = \frac{1}{48\pi}$$

Hence, when the height of the sand cone is 4 cm, its height is increasing at the rate of $\frac{1}{48\pi}$ cm/s.

15. The total cost C (x) in Rupees associated with the production of x units of an item is given by

$$C(x) = 0.007x^3 - 0.003x^2 + 15x + 4000.$$

ANS:

Marginal cost is the rate of change of total cost with respect to output.

$\therefore$Marginal cost (MC) $= \frac{dC}{dx} = 0.007(3x^2) - 0.003(2x) + 15$

$= 0.021x^2 - 0.006x + 15$

When $x = 17$, MC $= 0.021\ (17^2) - 0.006\ (17) + 15$

$= 0.021(289) - 0.006(17) + 15$

$= 6.069 - 0.102 + 15$

$= 20.967$

Hence, when 17 units are produced, the marginal cost is Rs. 20.967.

16. The total revenue in Rupees received from the sale of x units of a product given by

$$R(x) = 13x^2 + 26x + 15.$$

Find the marginal revenue when $x = 7$.

ANS:

Marginal revenue is the rate of change of total revenue with respect to the number of units sold.

$\therefore$Marginal Revenue (MR) $= \frac{dR}{dx} = 13(2x) + 26 = 26x + 26$

When $x = 7$,

MR $= 26(7) + 26 = 182 + 26 = 208$

Hence, the required marginal revenue is Rs 208.

17. The rate of change of the area of a circle with respect to its radius r at $r = 6$ cm is

(A) 10π (B) 12π (C) 8π (D) 11π

ANS:

The area of a circle (A) with radius (r) is given by,

$A = \pi r^2$

Therefore, the rate of change of the area with respect to its radius r is

$$\frac{dA}{dr} = \frac{d}{dr}\left(\pi r^2\right) = 2\pi r.$$

$\therefore$ When $r = 6$ cm,

$$\frac{dA}{dr} = 2\pi \times 6 = 12\pi \text{ cm}^2/\text{s}$$

Hence, the required rate of change of the area of a circle is 12π cm^2/s.

The correct answer is B.

18. The total revenue in Rupees received from the sale of x units of a product is given by
$R(x) = 3x^2 + 36x + 5$. The marginal revenue, when $x = 15$ is
(A) 116 (B) 96 (C) 90 (D) 126

ANS:

Marginal revenue is the rate of change of total revenue with respect to the number of units sold.

$\therefore$ Marginal Revenue (MR) $= \frac{dR}{dx} = 3(2x) + 36 = 6x + 36$

$\therefore$ When $x = 15$,

MR $= 6(15) + 36 = 90 + 36 = 126$

Hence, the required marginal revenue is Rs 126.

The correct answer is D.