CRPF PUBLIC SCHOOL, ROHINI

SECOND Intra School Mathematics Olympiad 2011

CLASS XII

SOLUTIONS

SECTION – A

1. (D) 91

Top layer has 1 orange
2^{nd} layer has 4 oranges
3^{rd} layer has 9 oranges
4^{th} layer has 16 oranges
5^{th} layer has 25 oranges
6^{th} layer has 36 oranges
Thus total is $1+4+9+16+25+36=91$

2. (C) 48

$125^{20}\times 16^{12}$

$=5^{3^{20}}\times 2^{4^{12}}$

$=5^{60}\times 2^{48}$

$=\left(5^{48}\times 2^{48}\right)\times 5^{12}$

$=5^{12}\times 10^{48}$

Clearly, 48 zeroes will come.

3. (B) 28

Solution 1

Since $PQRS$ is a square and $QR = 2 + 9 = 11$, then $PQ = QR = SR = PS = 11$.
The height of the shaded rectangle equals the height of the top left rectangle minus the height of the top right rectangle, or $6 - 2 = 4$.
The width of the shaded rectangle equals the width of the top right rectangle minus the width of the bottom right rectangle.
Since $SR = 11$, then the width of the bottom right rectangle is $11 - 10 = 1$.
Therefore, the width of the shaded rectangle is $8 - 1 = 7$.
Thus, the area of the shaded rectangle is $4 \times 7 = 28$.

Solution 2

Since $PQRS$ is a square and $QR = 2 + 9 = 11$, then $PQ = QR = SR = PS = 11$.
Since the side length of the square is 11, then its area is $11^2 = 121$.
Since $PQ = 11$, then the width of the top left rectangle is $11 - 8 = 3$, and so its area is $3 \times 6 = 18$.
Since $PS = 11$, then the height of the bottom left rectangle is $11 - 6 = 5$, and so its area is $5 \times 10 = 50$.
Since $SR = 11$, then the width of the bottom right rectangle is $11 - 10 = 1$, and so its area is $1 \times 9 = 9$.
The area of the top right rectangle is $8 \times 2 = 16$.
Thus, the area of the shaded rectangle equals the area of square $PQRS$ minus the combined areas of the four unshaded rectangles, or $121 - 18 - 50 - 9 - 16 = 28$.

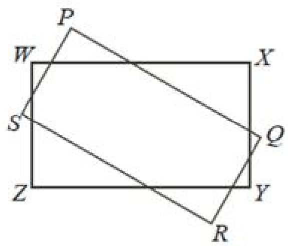
ANSWER: (B)

4. (D) 3

Note that $y = 0$ if and only if one of the factors equals 0. For the three factors, $b^2 - 4ac$ equals -11, 0, and 13, so that they have respectively 0, 1, and 2 roots.

5. (D) 8

Consider rectangles $WXYZ$ and $PQRS$.
Each of the four sides of $PQRS$ can intersect at most 2 of the sides of $WXYZ$, as any straight line can intersect at most two sides of a rectangle.
Therefore, the maximum possible number of points of intersection between the two rectangles is 8.
8 points of intersection is possible, as we can see in the diagram:



So the maximum possible number of points of intersection is 8.

6. (D) $\frac{2}{5}$

We consider choosing the three numbers all at once.
We list the possible sets of three numbers that can be chosen:

$\{1,2,3\}$ $\{1,2,4\}$ $\{1,2,5\}$ $\{1,3,4\}$ $\{1,3,5\}$ $\{1,4,5\}$ $\{2,3,4\}$ $\{2,3,5\}$ $\{2,4,5\}$ $\{3,4,5\}$

We have listed each in increasing order because once the numbers are chosen, we arrange them in increasing order.
There are 10 sets of three numbers that can be chosen.
Of these 10, the 4 sequences $1,2,3$ and $1,3,5$ and $2,3,4$ and $3,4,5$ are arithmetic sequences.
Therefore, the probability that the resulting sequence is an arithmetic sequence is $\frac{4}{10}$ or $\frac{2}{5}$.

7. (A) $\frac{100}{201}$

$$\sum_{i=1}^{100} \frac{1}{(2i-1)(2i+1)} = \sum_{i=1}^{100} \frac{1}{2}\left(\frac{1}{2i-1} - \frac{1}{2i+1}\right) = \frac{1}{2}\left(1 - \frac{1}{201}\right) = \frac{100}{201},$$

since all terms between 1 and $\frac{1}{201}$ occur with both positive and negative signs.

8. (D) $\frac{5}{4}$

Rewrite the equation as:

$$\sqrt{x} - \sqrt{1 - \frac{1}{x}} = \sqrt{x - \frac{1}{x}}$$

Now square both sides and simplify:

$$x - 2\sqrt{x-1} + 1 - \frac{1}{x} = x - \frac{1}{x}$$

$2\sqrt{x-1} = 1$ From here, it's simple to work out that x = 5/4.

9. (E) 524

We wish to calculate $f(f(2, f(3,4)), 5)$:

$$f(3,4) = 3 \cdot 4 + 2 \cdot 3 + 4 + 1 = 23$$
$$f(2,23) = 2 \cdot 23 + 2 \cdot 2 + 23 + 1 = 74$$
$$f(74,5) = 74 \cdot 5 + 2 \cdot 74 + 5 + 1 = 524$$

10. (C) 40

$2^{2^{3^{2^2}}} = 2^{2^{3^4}} = 2^{2^{81}} = 4^{4^x} = (2^2)^{(2^2)^x} = 2^{2(2^{2x})} = 2^{2^{1+2x}}$. It follows that $81 = 1+2x$ and that $x = 40$.

11. (B) 649

$$\begin{aligned} 2.5081081081081\ldots &= \frac{25.081081081\ldots}{10} = \frac{25 + \frac{81}{999}}{10} \\ &= \frac{5}{2} + \frac{3}{370} = \frac{5 \cdot 185 + 3}{370} = \frac{928}{370} = \frac{464}{185}. \end{aligned}$$

So the answer is $464 + 185 = 649$.

12. (C) 4

Solution 1

The centre of the circle is $(3, 0)$ and the circle has a radius of 5.

Thus $\sqrt{d^2 + 3^2} = 5$

$d^2 = 5^2 - 3^2$

$d^2 = 16$

Therefore $d = 4$, since $d > 0$.

Solution 2

Since AB is a diameter of the circle, $\angle ADB = 90°$ and $\angle AOD = 90°$.

$\Delta ADO \sim \Delta DBO$

Therefore, $\frac{OD}{AO} = \frac{BO}{OD}$

and $d^2 = 2(8)$

$d^2 = 16$

$d = 4$, since $d > 0$.

13. (A) 46

There are 64 cubes to start.

If we look at the bottom layer of cubes, we see that there are 6 uncovered cubes, each of which is missing 3 cubes above it. These are the only cubes that are missing.

Thus, there are $6(3) = 18$ missing cubes, so there are $64 - 18 = 46$ cubes remaining.

14. (E) 40

Because $1+2+\cdots+k=\frac{k(k+1)}{2}$, we have

$$\frac{1}{1+2+\cdots+k}=\frac{2}{k(k+1)}=\frac{2}{k}-\frac{2}{k+1}.$$

Hence,

$$\begin{aligned}\frac{1}{1+2}+\frac{1}{1+2+3}+\cdots+\frac{1}{1+2+\cdots+20} &= \frac{2}{2}-\frac{2}{3}+\frac{2}{3}-\frac{2}{4}+\cdots+\frac{2}{20}-\frac{2}{21}\\ &= 1-\frac{2}{21}=\frac{19}{21}\end{aligned}$$

and so $m+n=40$.

15. (A) 0

Since $x^2 \geq 0$ for all x, $x^2+5>0$. Since $(x^2-3)(x^2+5)<0$, $x^2-3<0$, so $x^2<3$ or $-\sqrt{3}<x<\sqrt{3}$. Thus $x=-1,0,1$.

SECTION – B

16.

Solution 1

A parabola is symmetric about its axis of symmetry.

Since the x-intercepts of the given parabola are $x=-1$ and $x=4$, then the axis of symmetry of the parabola is $x=\frac{-1+4}{2}=\frac{3}{2}$.

Since the point $(3,w)$ is $\frac{3}{2}$ units to the right of the axis of symmetry, then its y-coordinate (namely w) equals the y-coordinate of the point $\frac{3}{2}$ units to the left of the axis of symmetry, which is the point with $x=0$.

When $x=0$, we know that $y=8$.

Therefore, $w=8$.

(We could also note that $x=3$ is 1 unit to the left of the rightmost x-intercept so its y-coordinate is equal to that of the point 1 unit to the right of the leftmost x-intercept.)

Solution 2

Since the parabola has x-intercepts of -1 and 4, then its equation is of the form $y=a(x+1)(x-4)$ for some value of a.

Since the point $(0,8)$ lies on the parabola, then $8=a(1)(-4)$ or $a=-2$.

Therefore, the parabola has equation $y=-2(x+1)(x-4)$.

Since the point $(3,w)$ lies on the parabola, then $w=-2(4)(-1)=8$.

17.

Setting $\tan\frac{x}{2}=t$, we have $\sin x=\frac{2t}{1+t^2}$, $\cos x=\frac{1-t^2}{1+t^2}$. Then

$$\frac{6t}{1+t^2}+\frac{4(1-t^2)}{1+t^2}=5$$

which reduces to $9t^2-6t+1=0$ or, equivalently, $(3t-1)^2=0$. It follows that $t=\frac{1}{3}$ and

$$\tan x=\frac{2t}{1-t^2}=\frac{\frac{2}{3}}{\frac{8}{9}}=\frac{3}{4}$$

Then

$$2\sin x+\cos x+4\tan x=\frac{\frac{4}{3}}{\frac{10}{9}}+\frac{\frac{8}{9}}{\frac{10}{9}}+3=2+3=5$$

18.

The smallest two digit number is ten. You can add to ten any natural number up to 89 to get a two digit answer, so there are $89 - 9 = 80$ two digit numbers that can be added to ten to give a two digit answer. Similarly, there are $88 - 9 = 79$ numbers that can be added to eleven to give a two digit answer. There are 78 numbers that can be added to twelve, 77 that can be added to thirteen, and so forth through the one two digit number that can be added to 89. Each of these results in a new problem, so there are $80 + 79 + 78 + \cdots + 1$ distinct addition problems. This adds to $\frac{80(81)}{2} = 3240$.

19.

Let $f(x) = x^{200} - 2x^{199} + x^{50} - 2x^{49} + x^2 + x + 1$. Observe that $f(1) = 1$ and $f(2) = 7$. Let $q(x)$ and $r(x)$ denote the quotient and remainder, respectively, when $f(x)$ is divided by $(x-1)(x-2)$. Since $(x-1)(x-2)$ is a quadratic, we deduce that $r(x) = Ax + B$ for some numbers A and B. Thus,

$$f(x) = (x-1)(x-2)q(x) + Ax + B.$$

We deduce that $f(1) = A + B$ and $f(2) = 2A + B$. Since $f(1) = 1$ and $f(2) = 7$, we obtain $A + B = 1$ and $2A + B = 7$. We deduce that $A = 6$ and $B = -5$. Hence, $r(x) = 6x - 5$.

20.

Solution 1

Subtracting,

$$\begin{array}{r} x^2 - xy + 8 = 0 \\ x^2 - 8x + y = 0 \\ \hline -xy + 8x + 8 - y = 0 \\ 8(1+x) - y(1+x) = 0 \\ (8-y)(1+x) = 0 \\ y = 8 \quad \text{or} \quad x = -1 \end{array}$$

If $y = 8$, both equations become $x^2 - 8x + 8 = 0$, $x = 4 \pm 2\sqrt{2}$.
If $x = -1$ both equations become $y + 9 = 0$, $y = -9$.
The solutions are $(-1, -9)$, $\left(4 + 2\sqrt{2}, 8\right)$ and $\left(4 - 2\sqrt{2}, 8\right)$.

Solution 2

If $x^2 - xy + 8 = 0$, $y = \frac{x^2 + 8}{x}$.

And $x^2 - 8x + y = 0$ implies $y = 8x - x^2$.

Equating, $\frac{x^2 + 8}{x} = 8x - x^2$

or, $x^3 - 7x^2 + 8 = 0$.

By inspection, $x = -1$ is a root.
By division, $x^3 - 7x^2 + 8 = (x+1)\left(x^2 - 8x + 8\right)$.
As before, the solutions are $(-1, -9)$, $\left(4 \pm 2\sqrt{2}, 8\right)$.