CRPF PUBLIC SCHOOL, ROHINI

SECOND Intra School Mathematics Olympiad 2011

CLASS X

SOLUTIONS

A1)

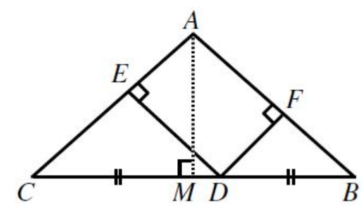
Solution

If we multiply the equation of the first line by 5 and the second by 7 we obtain, $5px + 10y = 35$ and $21x + 7qy = 35$. Comparing coefficients gives, $5p = 21$ or $p = \frac{21}{5}$. ANSWER: (D)

A2)

Solution

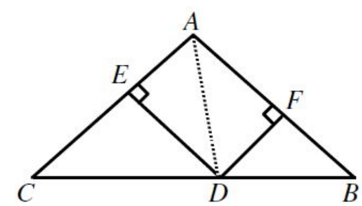
We start by drawing a line from A that is perpendicular to the base CB. Since ΔABC is isosceles, M is the midpoint of CB thus making $CM = MB = 20$. Using pythagoras in ΔACM we find AM to be $\sqrt{25^2 - 20^2} = 15$.



Join A to D. The area of ΔABC is $\frac{1}{2}(40)(15) = 300$ but it is also, $\frac{1}{2}(ED)(25) + \frac{1}{2}(DF)(25)$

$$= \frac{25}{2}(ED + DF).$$

Therefore, $ED + DF = \frac{2}{25}(300) = 24$.



ANSWER: (C)

A3)

Solution

A zero at the end of a number results from the product of 2 and 5. The number of zeros at the end of a number equals the number of product pairs of 2 and 5 that can be formed from the prime factorization of that number.

Since $15^6 \times 28^5 \times 55^7 = (3.5)^6 \left(2^2.7\right)^5 (5.11)^7$

$$= 3^6.5^6.2^{10}.7^5.5^7.11^7$$
$$= 3^6.5^3.7^5.11^7.10^{10}$$

There will be ten zeros in the string at the end of the number.

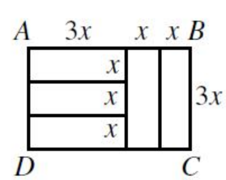
ANSWER: (A)

A4)

Solution

If we let the width of each rectangle be x units then the length of each rectangle is $3x$ units. (This is illustrated in the diagram.) The length, AB, is now $3x + x + x$ or $5x$ units and $BC = 3x$. Thus $AB : BC = 5x : 3x$

$$= 5:3, \quad x \neq 0.$$

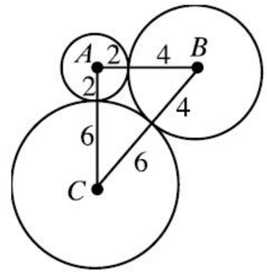


ANSWER: (D)

A5)

Solution

Since the circles are mutually tangent, the lines joining their centres pass through the points of tangency. Thus, the sides of ΔABC have lengths 6, 8 and 10 if we write in the radii as shown.
Since $10^2 = 6^2 + 8^2$, the triangle is right-angled at A.



ANSWER: (C)

A6)

Solution

From the equalities $5a = 4b = 3c = 2d = e$, we conclude that e is the largest and a is the smallest of the integers. Since e is divisible by 5, 4, 3, and 2, the smallest possible value for e is 60. The corresponding values for a, b, c, and d, are 12, 15, 20, and 30, respectively. Thus, the smallest positive integer value for k is $12 + 2(15) + 3(20) + 4(30) + 5(60) = 522$.

A7)

Solution

The area of ΔMNC is $\frac{1}{2}(1)(1) = \frac{1}{2}$. Since ΔBDC is half the square, it will have an area of 2.
Since the shaded region has an area equal to that of ΔBDC minus the area of ΔMNC, its area will be $2 - \frac{1}{2} = \frac{3}{2}$.

ANSWER: (D)

A8)

Solution

The units digit of $(2002)^{2002}$ is the same as the units digit of 2^{2002}, since the first three digits of 2002 do not affect the units digit.
We write out the first few powers of 2 and check for the units digit.

n	1	2	3	4	5	6	7	8	9
2^n	2	4	8	16	32	64	128	256	512

From this table, we see that the units digits repeat every 4 powers. So the units digit of 2^{2000} will be 6, and thus the units digit of 2^{2002} (and so also of $(2002)^{2002}$) will be 4.

ANSWER: (A)

A9)

We determine the smallest and largest three-digit perfect squares first.
Clearly, the smallest three-digit perfect square is $100 = 10^2$.
Now $\sqrt{1000} \approx 31.6$, so the largest perfect square less than 1000 is $31^2 = 961$ (since $31^2 < 1000$ and $32^2 > 1000$).
Therefore, only the perfect squares between $10^2 = 100$ and $31^2 = 961$ have three digits, which means that exactly 22 three-digit positive integers are perfect squares.

ANSWER: (B)

A10}

Using the rules for forming the sequence, the third term is $a+2$ and the fourth term is $a+2+(a+2)=2(a+2)$. Similarly, the fifth term is $4(a+2)$ and the sixth term is $8(a+2)$. But the sixth term is 56, so $8(a+2)=56$ or $a+2=7$ or $a=5$.

ANSWER: (E)

A11)

Since x is between -1 and 0, then x^2 is between 0 and 1, and so $-x^2$ is between -1 and 0. Therefore, the best letter is either b or c. When a number between -1 and 0 is squared, it becomes closer to 0 than it was before, so the best choice must be c, not b.

Answer: (C)

A12)

Since the seven children were born in seven consecutive years, then the oldest child is 4 years older than the oldest of the three youngest children, the second oldest child is 4 years older than the second oldest of the three youngest children, and the third oldest child is 4 years older than the youngest child.

Since the sum of the ages of the three youngest children is 42, then the sum of the ages of the three oldest children is $42+4+4+4=54$.

Solution 2

Since the ages of the seven children are seven consecutive integers, let the ages of the youngest three children be x, $x+1$ and $x+2$.

Then $x+x+1+x+2=42$ or $3x+3=42$ or $x=13$.

So the ages of the seven children are 13, 14, 15, 16, 17, 18, and 19.

Therefore, the sum of the ages of the oldest three children is $17+18+19=54$.

ANSWER: (B)

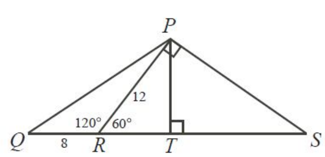
A13)

A score of 11 points can be obtained with 3 correct, 2 unanswered and 5 wrong.
A score of 13 points can be obtained with 4 correct, 1 unanswered and 5 wrong.
A score of 17 points can be obtained with 5 correct, 2 unanswered and 3 wrong.
A score of 23 points can be obtained with 7 correct, 2 unanswered and 1 wrong.
Therefore, by process of elimination, 29 is the total score which is not possible.
(Why is 29 not possible? If all ten questions are correct, the total score would be 30 points. If 9 or fewer questions are correct, at least 2 points will be lost from the maximum possible, ie. the maximum possible score is 28. Therefore, 29 is not possible.)

ANSWER: (E)

A14)

Since $\angle QRP = 120°$ and QRS is a straight line, then $\angle PRS = 180° - 120° = 60°$.
Since $\angle RPS = 90°$, then $\triangle SRP$ is a 30°-60°-90° triangle.
Therefore, $RS = 2PR = 2(12) = 24$.
Drop a perpendicular from P to T on RS.



Since $\angle PRT = 60°$ and $\angle PTR = 90°$, then $\triangle PRT$ is also a 30°-60°-90° triangle.
Therefore, $PT = \frac{\sqrt{3}}{2}PR = 6\sqrt{3}$.
Consider $\triangle QPS$. We may consider QS as its base with height PT.
Thus, its area is $\frac{1}{2}(6\sqrt{3})(8+24) = 96\sqrt{3}$.

ANSWER: (E)

A15)

A graph that is linear with a slope of 0 is a horizontal straight line. This is Graph Q.

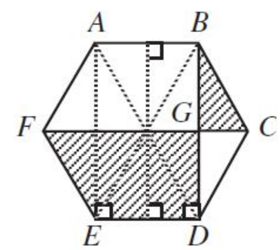
ANSWER: (B)

A16)

Solution 1

Join E to B and D to A as shown. Also join E to A and draw a line parallel to AE through the point of intersection of BE and AD. Quadrilateral $FEDG$ is now made up of five triangles each of which has the same area as ΔBCG.

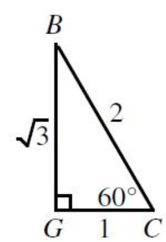
The required ratio is 5:1.



or

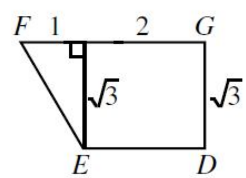
Solution 2

For convenience, assume that each side of the hexagon has a length of 2 units. Each angle in the hexagon equals $120°$ so $\angle BCG = \frac{1}{2}(120°) = 60°$. Now label ΔBCG as shown. Using the standard ratios for a $30° - 60° - 90°$ triangle we have $BG = \sqrt{3}$ and $GC = 1$.



The area of $\Delta BCG = \frac{1}{2}(1)\sqrt{3} = \frac{\sqrt{3}}{2}$. Dividing the quadrilateral $FGDE$ as illustrated, it will have an area of $2(\sqrt{3}) + \frac{1}{2}(1)(\sqrt{3}) = \frac{5\sqrt{3}}{2}$.

The required ratio is $\frac{5\sqrt{3}}{2} : \frac{\sqrt{3}}{2}$ or 5:1, as in solution 1.



ANSWER: (E)

A17)

Solution

Consider the system of equations $x^2 + x^2y^2 + x^2y^4 = 525 \quad (1)$

and $x + xy + xy^2 = 35 \quad (2)$

The expression on the left side of equation (1) can be rewritten as,

$$x^2 + x^2y^2 + x^2y^4 = \left(x + xy^2\right)^2 - x^2y^2$$
$$= \left(x + xy^2 - xy\right)\left(x + xy^2 + xy\right)$$

Thus, $\left(x + xy^2 - xy\right)\left(x + xy^2 + xy\right) = 525$

Substituting from (2) gives, $\left(x + xy^2 - xy\right)(35) = 525$

or, $x + xy^2 - xy = 15 \quad (3)$

Now subtracting (3) from (2), $2xy = 20,\ x = \frac{10}{y}$

Substituting for x in (3) gives,

$$\frac{10}{y} + 10y - 10 = 15$$

$$10y^2 - 25y + 10 = 0$$
$$2y^2 - 5y + 2 = 0$$
$$(2y-1)(y-2) = 0$$
$$y = \frac{1}{2} \text{ or } y = 2$$

The sum of the real y values satisfying the system is $\frac{5}{2}$.

ANSWER: (E)

Since the distance from Cindy's home to school is unknown, represent this distance by d, in kilometres. We will consider the problem in two separate cases, the first in which she travels at 20 km/h and the second when she travels at 10 km/h.

Distance travelled at 20 km/h = Distance travelled at 10 km/h

Let the time that Cindy takes travelling home at 20 km/h be t hours.

If Cindy arrives home $\frac{3}{4}$ h later when travelling at 10 km/h, then the length of time travelling is $\left(t + \frac{3}{4}\right)$ hours. The previous equation becomes

$$20t = 10\left(t + \frac{3}{4}\right)$$
$$20t = 10t + \frac{30}{4}$$
$$10t = \frac{15}{2}$$

A18)

$$t = \frac{15}{20} \text{ or } \frac{3}{4}.$$

Therefore the distance from school to home is $d = 20 \times \frac{3}{4}$, or $d = 15$ km.

If Cindy arrives home at 5:00 in the afternoon, she would have travelled home in $\frac{3}{4} + \frac{1}{2} = \frac{5}{4}$ hours over a distance of 15 kilometres.

Therefore, $s = \frac{d}{t} = \frac{15}{\frac{5}{4}} = 15 \times \frac{4}{5} = 12$ km/h.

Therefore, Cindy would have had to travel at 12 km/h to arrive home at 5:00 p.m.

ANSWER: (D)

A19)

When a number n is divided by x, the remainder is the difference between n and the largest multiple of x less than n. When 100 is divided by a positive integer x, the remainder is 10. This means that $100 - 10 = 90$ is exactly divisible by x. It also means that x is larger than 10, otherwise the remainder would be smaller than 10.
Since 90 is exactly divisible by x, then $11 \times 90 = 990$ is also exactly divisible by x.
Since $x > 10$, then the next multiple of x is $990 + x$, which is larger than 1000.
Thus, 990 is the largest multiple of x less than 1000, and so the remainder when 1000 is divided by x is $1000 - 990 = 10$.

Solution 2
When 100 is divided by a positive integer x, the remainder is 10. The remainder is the difference between 100 and the largest multiple of x less than 100.
Therefore, the largest multiple of x less than 100 is $100 - 10 = 90$.
It also means that x is larger than 10, otherwise the remainder would be smaller than 10.
We can choose $x = 15$, since $6 \times 15 = 90$ and 15 is larger than 10.
What is the remainder when 1000 is divided by 15? Using a calculator, $1000 \div 15 = 66.666\ldots$ and $66 \times 15 = 990$.
Thus, the difference between 1000 and the largest multiple of 15 less than 1000 (that is, 990) equals 10 and so the remainder is equal to 10.

ANSWER: (A)

A20)

Since $5K3L - M4N1 = 4451$, then

$$\begin{array}{cccc} & M & 4 & N & 1 \\ + & 4 & 4 & 5 & 1 \\ \hline & 5 & K & 3 & L \end{array}$$

We start from the units column and work towards the left.
Considering the units column, the sum $1 + 1$ has a units digit of L. Thus, $L = 2$. (There is no carry to the tens column.)
Considering the tens column, the sum $N + 5$ has a units digit of 3. Thus, $N = 8$. (There is a carry of 1 to the hundreds column.) This gives

$$\begin{array}{ccccc} & & ^{1} & & \\ & M & 4 & 8 & 1 \\ + & 4 & 4 & 5 & 1 \\ \hline & 5 & K & 3 & 2 \end{array}$$

Considering the hundreds column, the sum $4 + 4$ plus the carry of 1 from the tens column has a units digit of K. Therefore, $K = 4 + 4 + 1 = 9$. There is no carry from the hundreds column to the thousands column.
Considering the thousands column, the sum $M + 4$ equals 5. Therefore, $M = 1$. This gives

$$\begin{array}{ccccc} & 1 & 4 & 8 & 1 \\ + & 4 & 4 & 5 & 1 \\ \hline & 5 & 9 & 3 & 2 \end{array}$$

which is equivalent to $5932 - 4451 = 1481$, which is correct.
Finally, $K + L + M + N = 9 + 2 + 1 + 8 = 20$.

ANSWER: (A)