CLASS XII

CHAPTER 5 CONTINUITY AND DIFFERNENTIABILITY

NCERT EX 5.8 SOLUTIONS

Question 1:

Verify Rolle's Theorem for the function $f(x)=x^2+2x-8$, $x\in[-4,2]$

ANS:

The given function, $f(x)=x^2+2x-8$, being a polynomial function, is continuous in [−4, 2] and is differentiable in (−4, 2).

$$f(-4)=(-4)^2+2\times(-4)-8=16-8-8=0$$
$$f(2)=(2)^2+2\times 2-8=4+4-8=0$$

$\therefore f(-4)=f(2)=0$

$\Rightarrow$ The value of $f(x)$ at −4 and 2 coincides.

Rolle's Theorem states that there is a point $c\in(-4,2)$ such that $f'(c)=0$

$$f(x)=x^2+2x-8$$
$$\Rightarrow f'(x)=2x+2$$
$$\therefore f'(c)=0$$
$$\Rightarrow 2c+2=0$$
$$\Rightarrow c=-1,\ \text{where } c=-1\in(-4,2)$$

Hence, Rolle's Theorem is verified for the given function.

Question 2:

Examine if Rolle's Theorem is applicable to any of the following functions. Can you say some thing about the converse of Rolle's Theorem from these examples?

(i) $f(x)=[x]$ for $x\in[5,\ 9]$

(ii) $f(x)=[x]$ for $x\in[-2,\ 2]$

(iii) $f(x)=x^2-1$ for $x\in[1,\ 2]$

ANS:

(i) $f(x)=[x]$ for $x \in [5,\ 9]$

It is evident that the given function $f(x)$ is not continuous at every integral point.

In particular, $f(x)$ is not continuous at $x = 5$ and $x = 9$

$\Rightarrow f(x)$ is not continuous in $[5, 9]$.

Also, $f(5)=[5]=5$ and $f(9)=[9]=9$

$\therefore f(5) \neq f(9)$

The differentiability of f in $(5, 9)$ is checked as follows.

Let n be an integer such that $n \in (5, 9)$.

The left hand limit of f at $x = n$ is,

$$\lim_{h\to 0^-} \frac{f(n+h)-f(n)}{h} = \lim_{h\to 0^-} \frac{[n+h]-[n]}{h} = \lim_{h\to 0^-} \frac{n-1-n}{h} = \lim_{h\to 0^-} \frac{-1}{h} = \infty$$

The right hand limit of f at $x = n$ is,

$$\lim_{h\to 0^+} \frac{f(n+h)-f(n)}{h} = \lim_{h\to 0^+} \frac{[n+h]-[n]}{h} = \lim_{h\to 0^+} \frac{n-n}{h} = \lim_{h\to 0^+} 0 = 0$$

Since the left and right hand limits of f at $x = n$ are not equal, f is not differentiable at $x = n$

$\therefore f$ is not differentiable in $(5, 9)$.

It is observed that f does not satisfy all the conditions of the hypothesis of Rolle's Theorem.

Hence, Rolle's Theorem is not applicable for $f(x)=[x]$ for $x \in [5,\ 9]$.

(ii) $f(x)=[x]$ for $x \in [-2,\ 2]$

It is evident that the given function $f(x)$ is not continuous at every integral point.

In particular, $f(x)$ is not continuous at $x = -2$ and $x = 2$

$\Rightarrow f(x)$ is not continuous in $[-2, 2]$.

Also, $f(-2)=[-2]=-2$ and $f(2)=[2]=2$

$\therefore f(-2) \neq f(2)$

The differentiability of f in $(-2, 2)$ is checked as follows.

Let n be an integer such that $n \in (-2, 2)$.

The left hand limit of f at $x = n$ is,

$$\lim_{h\to 0^-} \frac{f(n+h)-f(n)}{h} = \lim_{h\to 0^-} \frac{[n+h]-[n]}{h} = \lim_{h\to 0^-} \frac{n-1-n}{h} = \lim_{h\to 0^-} \frac{-1}{h} = \infty$$

The right hand limit of f at $x = n$ is,

$$\lim_{h\to 0^+} \frac{f(n+h)-f(n)}{h} = \lim_{h\to 0^+} \frac{[n+h]-[n]}{h} = \lim_{h\to 0^+} \frac{n-n}{h} = \lim_{h\to 0^+} 0 = 0$$

Since the left and right hand limits of f at $x = n$ are not equal, f is not differentiable at $x = n$

$\therefore f$ is not differentiable in $(-2, 2)$.

It is observed that f does not satisfy all the conditions of the hypothesis of Rolle's Theorem.

Hence, Rolle's Theorem is not applicable for $f(x) = [x]$ for $x \in [-2, 2]$.

(iii) $f(x) = x^2 - 1$ for $x \in [1, 2]$

It is evident that f, being a polynomial function, is continuous in $[1, 2]$ and is differentiable in $(1, 2)$.

$$f(1) = (1)^2 - 1 = 0$$
$$f(2) = (2)^2 - 1 = 3$$

$\therefore f(1) \neq f(2)$

It is observed that f does not satisfy a condition of the hypothesis of Rolle's Theorem.

Hence, Rolle's Theorem is not applicable for $f(x) = x^2 - 1$ for $x \in [1, 2]$.

Question 3:

If $f : [-5, 5] \to \mathbf{R}$ is a differentiable function and if $f'(x)$ does not vanish anywhere, then prove that $f(-5) \neq f(5)$.

ANS:

It is given that $f:[-5,5]\rightarrow \mathbf{R}$ is a differentiable function.

Since every differentiable function is a continuous function, we obtain

(a) f is continuous on $[-5, 5]$.

(b) f is differentiable on $(-5, 5)$.

Therefore, by the Mean Value Theorem, there exists $c \in (-5, 5)$ such that

$$f'(c)=\frac{f(5)-f(-5)}{5-(-5)}$$
$$\Rightarrow 10f'(c)=f(5)-f(-5)$$

It is also given that $f'(x)$ does not vanish anywhere.

$$\therefore f'(c)\neq 0$$
$$\Rightarrow 10f'(c)\neq 0$$
$$\Rightarrow f(5)-f(-5)\neq 0$$
$$\Rightarrow f(5)\neq f(-5)$$

Hence, proved.

Question 4:

Verify Mean Value Theorem, if $f(x)=x^2-4x-3$ in the interval $[a,\ b]$, where $a=1$ and $b=4$.

ANS:

The given function is $f(x)=x^2-4x-3$

f, being a polynomial function, is continuous in $[1, 4]$ and is differentiable in $(1, 4)$ whose derivative is $2x-4$.

$$f(1)=1^2-4\times1-3=-6,\ f(4)=4^2-4\times4-3=-3$$
$$\therefore \frac{f(b)-f(a)}{b-a}=\frac{f(4)-f(1)}{4-1}=\frac{-3-(-6)}{3}=\frac{3}{3}=1$$

Mean Value Theorem states that there is a point $c \in (1, 4)$ such that $f'(c)=1$

$$f'(c)=1$$
$$\Rightarrow 2c-4=1$$
$$\Rightarrow c=\frac{5}{2},\text{ where } c=\frac{5}{2}\in(1,\ 4)$$

Hence, Mean Value Theorem is verified for the given function.

Question 5:

Verify Mean Value Theorem, if $f(x) = x^3 - 5x^2 - 3x$ in the interval $[a, b]$, where $a = 1$ and $b = 3$. Find all $c \in (1,3)$ for which $f'(c) = 0$

ANS:

The given function *f* is $f(x) = x^3 - 5x^2 - 3x$

f, being a polynomial function, is continuous in [1, 3] and is differentiable in (1, 3) whose derivative is $3x^2 - 10x - 3$.

$$f(1) = 1^3 - 5\times1^2 - 3\times1 = -7,\ f(3) = 3^3 - 5\times3^2 - 3\times3 = -27$$
$$\therefore \frac{f(b) - f(a)}{b - a} = \frac{f(3) - f(1)}{3 - 1} = \frac{-27 - (-7)}{3 - 1} = -10$$

Mean Value Theorem states that there exist a point $c \in (1, 3)$ such that $f'(c) = -10$

$$f'(c) = -10$$
$$\Rightarrow 3c^2 - 10c - 3 = 10$$
$$\Rightarrow 3c^2 - 10c + 7 = 0$$
$$\Rightarrow 3c^2 - 3c - 7c + 7 = 0$$
$$\Rightarrow 3c(c - 1) - 7(c - 1) = 0$$
$$\Rightarrow (c - 1)(3c - 7) = 0$$
$$\Rightarrow c = 1,\ \frac{7}{3},\ \text{where } c = \frac{7}{3} \in (1,\ 3)$$

Hence, Mean Value Theorem is verified for the given function and $c = \frac{7}{3} \in (1,\ 3)$ is the only point for which $f'(c) = 0$

Question 6:

Examine the applicability of Mean Value Theorem for all three functions given in the above exercise 2.

ANS:

(i) $f(x)=[x]$ for $x \in [5,\ 9]$

It is evident that the given function $f(x)$ is not continuous at every integral point.

In particular, $f(x)$ is not continuous at $x = 5$ and $x = 9$

$\Rightarrow f(x)$ is not continuous in $[5, 9]$.

The differentiability of f in $(5, 9)$ is checked as follows.

Let n be an integer such that $n \in (5, 9)$.

The left hand limit of f at $x = n$ is,

$$\lim_{h \to 0^-} \frac{f(n+h)-f(n)}{h} = \lim_{h \to 0^-} \frac{[n+h]-[n]}{h} = \lim_{h \to 0^-} \frac{n-1-n}{h} = \lim_{h \to 0^-} \frac{-1}{h} = \infty$$

The right hand limit of f at $x = n$ is,

$$\lim_{h \to 0^+} \frac{f(n+h)-f(n)}{h} = \lim_{h \to 0^+} \frac{[n+h]-[n]}{h} = \lim_{h \to 0^+} \frac{n-n}{h} = \lim_{h \to 0^+} 0 = 0$$

Since the left and right hand limits of f at $x = n$ are not equal, f is not differentiable at $x = n$

$\therefore f$ is not differentiable in (5, 9).

It is observed that f does not satisfy all the conditions of the hypothesis of Mean Value Theorem.

Hence, Mean Value Theorem is not applicable for $f(x)=[x]$ for $x\in[5,\ 9]$.

(ii) $f(x)=[x]$ for $x\in[-2,\ 2]$

It is evident that the given function $f(x)$ is not continuous at every integral point.

In particular, $f(x)$ is not continuous at $x=-2$ and $x=2$

$\Rightarrow f(x)$ is not continuous in $[-2, 2]$.

The differentiability of f in $(-2, 2)$ is checked as follows.

Let n be an integer such that $n \in (-2, 2)$.

The left hand limit of f at $x=n$ is,

$$\lim_{h\to 0^-}\frac{f(n+h)-f(n)}{h}=\lim_{h\to 0^-}\frac{[n+h]-[n]}{h}=\lim_{h\to 0^-}\frac{n-1-n}{h}=\lim_{h\to 0^-}\frac{-1}{h}=\infty$$

The right hand limit of f at $x=n$ is,

$$\lim_{h\to 0^+}\frac{f(n+h)-f(n)}{h}=\lim_{h\to 0^+}\frac{[n+h]-[n]}{h}=\lim_{h\to 0^+}\frac{n-n}{h}=\lim_{h\to 0^+}0=0$$

Since the left and right hand limits of f at $x=n$ are not equal, f is not differentiable at $x=n$

$\therefore f$ is not differentiable in $(-2, 2)$.

It is observed that f does not satisfy all the conditions of the hypothesis of Mean Value Theorem.

Hence, Mean Value Theorem is not applicable for $f(x)=[x]$ for $x\in[-2,\ 2]$.

(iii) $f(x)=x^2-1$ for $x\in[1,\ 2]$

It is evident that f, being a polynomial function, is continuous in [1, 2] and is differentiable in (1, 2).

It is observed that f satisfies all the conditions of the hypothesis of Mean Value Theorem.

Hence, Mean Value Theorem is applicable for $f(x)=x^2-1$ for $x\in[1,\ 2]$.

It can be proved as follows.

$$f(1)=1^2-1=0,\ f(2)=2^2-1=3$$

$$\therefore \frac{f(b)-f(a)}{b-a}=\frac{f(2)-f(1)}{2-1}=\frac{3-0}{1}=3$$

$$f'(x)=2x$$

$$\therefore f'(c)=3$$

$$\Rightarrow 2c=3$$

$$\Rightarrow c=\frac{3}{2}=1.5,\ \text{where } 1.5\in[1, 2]$$