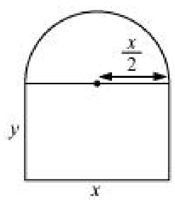
SOLUTIONS TO CLASS TEST XII C

WORD PROBLEMS (MAX/MIN.) HELD ON 23 AUG 2011

ANS 1.

Let x and y be the length and breadth of the rectangular window.

Radius of the semicircular opening $= \frac{x}{2}$



It is given that the perimeter of the window is 10 m.

$$\therefore x + 2y + \frac{\pi x}{2} = 10$$

$$\Rightarrow x\left(1+\frac{\pi}{2}\right) + 2y = 10$$

$$\Rightarrow 2y = 10 - x\left(1+\frac{\pi}{2}\right)$$

$$\Rightarrow y = 5 - x\left(\frac{1}{2}+\frac{\pi}{4}\right)$$

$\therefore$ Area of the window (A) is given by,

$$A = xy + \frac{\pi}{2}\left(\frac{x}{2}\right)^2$$

$$= x\left[5 - x\left(\frac{1}{2}+\frac{\pi}{4}\right)\right] + \frac{\pi}{8}x^2$$

$$= 5x - x^2\left(\frac{1}{2}+\frac{\pi}{4}\right) + \frac{\pi}{8}x^2$$

$$\therefore \frac{dA}{dx} = 5 - 2x\left(\frac{1}{2}+\frac{\pi}{4}\right) + \frac{\pi}{4}x$$

$$= 5 - x\left(1+\frac{\pi}{2}\right) + \frac{\pi}{4}x$$

$$\therefore \frac{d^2A}{dx^2} = -\left(1+\frac{\pi}{2}\right) + \frac{\pi}{4} = -1 - \frac{\pi}{4}$$

Now, $\frac{dA}{dx}=0$

$$\Rightarrow 5-x\left(1+\frac{\pi}{2}\right)+\frac{\pi}{4}x=0$$

$$\Rightarrow 5-x-\frac{\pi}{4}x=0$$

$$\Rightarrow x\left(1+\frac{\pi}{4}\right)=5$$

$$\Rightarrow x=\frac{5}{\left(1+\frac{\pi}{4}\right)}=\frac{20}{\pi+4}$$

Thus, when $x=\frac{20}{\pi+4}$ then $\frac{d^2A}{dx^2}<0.$

Therefore, by second derivative test, the area is the maximum when length $x=\frac{20}{\pi+4}$ m.

Now,

$$y=5-\frac{20}{\pi+4}\left(\frac{2+\pi}{4}\right)=5-\frac{5(2+\pi)}{\pi+4}=\frac{10}{\pi+4}\text{ m}$$

Hence, the required dimensions of the window to admit maximum light is given by length $=\frac{20}{\pi+4}$ m and breadth $=\frac{10}{\pi+4}$ m.

ANS 2.

Let a piece of length l be cut from the given wire to make a square.

Then, the other piece of wire to be made into a circle is of length $(28-l)$ m.

Now, side of square $=\frac{l}{4}$.

Let r be the radius of the circle. Then, $2\pi r=28-l \Rightarrow r=\frac{1}{2\pi}(28-l)$.

The combined areas of the square and the circle (A) is given by,

$$A=(\text{side of the square})^2+\pi r^2$$

$$=\frac{l^2}{16}+\pi\left[\frac{1}{2\pi}(28-l)\right]^2$$

$$=\frac{l^2}{16}+\frac{1}{4\pi}(28-l)^2$$

$$\therefore \frac{dA}{dl}=\frac{2l}{16}+\frac{2}{4\pi}(28-l)(-1)=\frac{l}{8}-\frac{1}{2\pi}(28-l)$$

$$\frac{d^2A}{dl^2}=\frac{1}{8}+\frac{1}{2\pi}>0$$

Now, $\frac{dA}{dl}=0 \Rightarrow \frac{l}{8}-\frac{1}{2\pi}(28-l)=0$

$$\Rightarrow \frac{\pi l-4(28-l)}{8\pi}=0$$

$$\Rightarrow (\pi+4)l-112=0$$

$$\Rightarrow l=\frac{112}{\pi+4}$$

Thus, when $l = \frac{112}{\pi+4}, \frac{d^2A}{dl^2} > 0.$

$\therefore$ By second derivative test, the area (A) is the minimum when $l = \frac{112}{\pi+4}$.

Hence, the combined area is the minimum when the length of the wire in making the square is $\frac{112}{\pi+4}$ cm while the length of the wire in making the circle is $28 - \frac{112}{\pi+4} = \frac{28\pi}{\pi+4}$ cm.

ANS3.

Let r and h be the radius and the height (altitude) of the cone respectively.

Then, the volume (V) of the cone is given as:

$$V = \frac{1}{3\pi}\pi r^2 h \Rightarrow h = \frac{3V}{r^2}$$

The surface area (S) of the cone is given by,

$S = \pi r l$ (where l is the slant height)

$$= \pi r\sqrt{r^2 + h^2}$$

$$= \pi r\sqrt{r^2 + \frac{9V^2}{\pi^2 r^4}} = \pi\frac{r\sqrt{\pi^2 r^6 + 9V^2}}{\pi r^2}$$

$$= \frac{1}{r}\sqrt{\pi^2 r^6 + 9V^2}$$

$$\therefore \frac{dS}{dr} = \frac{r \cdot \frac{6\pi^2 r^5}{2\sqrt{\pi^2 r^6 + 9V^2}} - \sqrt{\pi^2 r^6 + 9V^2}}{r^2}$$

$$= \frac{3\pi^2 r^6 - \pi^2 r^6 - 9V^2}{r^2\sqrt{\pi^2 r^6 + 9V^2}}$$

$$= \frac{2\pi^2 r^6 - 9V^2}{r^2\sqrt{\pi^2 r^6 + 9V^2}}$$

$$= \frac{2\pi^2 r^6 - 9V^2}{r^2\sqrt{\pi^2 r^6 + 9V^2}}$$

Now, $\frac{dS}{dr} = 0 \Rightarrow 2\pi^2 r^6 = 9V^2 \Rightarrow r^6 = \frac{9V^2}{2\pi^2}$

Thus, it can be easily verified that when $r^6 = \frac{9V^2}{2\pi^2}, \frac{d^2S}{dr^2} > 0.$

$\therefore$ By second derivative test, the surface area of the cone is the least when $r^6 = \frac{9V^2}{2\pi^2}.$

When $r^6 = \frac{9V^2}{2\pi^2}, \ h = \frac{3V}{\pi r^2} = \frac{3}{\pi r^2}\left(\frac{2\pi^2 r^6}{9}\right)^{\frac{1}{2}} = \frac{3}{\pi r^2} \cdot \frac{\sqrt{2}\pi r^3}{3} = \sqrt{2}r.$

Hence, for a given volume, the right circular cone of the least curved surface has an altitude equal to $\sqrt{2}$ times the radius of the base.

ANS 4. TRY YOURSELF